The *cross-ratio* of $p, q, r, s \in \hat{\mathbb{C}}$ is given by $[p, q, r, s] = \frac{(p-q)(r-s)}{(p-s)(r-q)}$.
Recall that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \;;\quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \;;\quad \cosh z = \frac{e^{z} + e^{-z}}{2} \;;\quad \sinh z = \frac{e^{z} - e^{-z}}{2}\,.$$

Four forms of the Cauchy-Riemann equations, where $z = x + iy = re^{i\theta}$ and $f(z) = u + iv = se^{i\alpha}$:

$$\left\{ \begin{array}{c} u_x = v_y, \\ u_y = -v_x \end{array} \right\} \quad \left\{ \begin{array}{c} s_x = s{\cdot}\alpha_y, \\ s_y = -s{\cdot}\alpha_x \end{array} \right\} \quad \left\{ \begin{array}{c} u_\theta = -r{\cdot}v_r, \\ v_\theta = r{\cdot}u_r \end{array} \right\} \quad \left\{ \begin{array}{c} -rs{\cdot}\alpha_r = s_\theta \\ r{\cdot}s_r = s{\cdot}\alpha_\theta \end{array} \right\}$$

(1) For each of the following parametrized curves, sketch the curve. Then find its tangent vector $\gamma'(t)$ at the start point and end point. Using the arc length formula, find the length of the curve.
(a) $\gamma(t) = e^{it}$, $t \in [1, 4]$
(b) $\gamma(t) = e^{it^2}$, $t \in [1, 2]$
(c) $\gamma(t) = \begin{cases} (t+1) + i(t^2 + 2t), & -2 \le t < 0 \\ 1 - t, & -2 \le 0 \le t \le 2 \end{cases}$

(2) (a) Find the points at which each of the following functions are differentiable. Are they analytic anywhere?
- $e^{-y}(\cos x + i \sin x)$ • $\cos x - i \sin y$ • $r^3 + 3i\theta$ • $\left[re^{r\cos\theta}\right]{\cdot}e^{i(\theta + r\cos\theta)}$

(b) If a curve passes through the point $z_0 = -\pi$ with a tangent line of slope m, what can you say about the image of that curve under the map $f(x + iy) = \cos x - i \sin y$?

(3) (a) What is the derivative of $f(z) = [z, -2, -1, \infty]$? Derive this computationally and give a geometric description of f.
(b) Write $T(z) = \frac{iz+1}{2z-1}$ as a cross-ratio map. That is, find the q, r, s such that $T(z) = [z, q, r, s]$.
(c) The function $f(z) = \displaystyle\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2 - 1)}}$ is a Schwarz-Christoffel transformation taking the upper half-plane conformally to the interior of a square. Give a modification whose image is a rectangle but not a square. Give a second modification whose image is a quadrilateral but not a rectangle.

(4) Some computations.
(a) Derive the trig identity $\sin(2z) = 2 \sin z \cos z$.
(b) Show that the derivative of tanh is sech^2. At what points is the hyperbolic tangent function angle-preserving?
(c) Compute all values of the log base i of 5. (Use principal values for logarithms.)

(5) Evaluate the following integrals, where S is a closed square with counterclockwise orientation and vertices at $\pm 2 \pm 2i$.
- $\displaystyle\oint_S \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$ • $\displaystyle\oint_S \frac{\cos z}{z(z^2 - 8)}\, dz$ • $\displaystyle\oint_S \frac{z}{(2z+1)^2}\, dz$ • $\displaystyle\oint_S z^n e^{1/z} dz$

(6) Let D be the closed disk of radius $r > 0$ centered at z_0. What is the maximum value of $|e^{3z}|$ on D and where is it attained?

(7) Given a circle $C = C_r(p)$ centered at p of radius r, let us say that *inversion in the circle* is the map $I_C : \mathbb{C} \setminus \{p\} \to \mathbb{C}$ with the following property: given any $z \in \mathbb{C}$, its image $I_C(z)$ is the unique point on the ray $\overrightarrow{pz}$ such that $|z - p| \cdot |I(z) - p| = r^2$.

(a) Prove that I_C has a continuous extension to all of $\hat{\mathbb{C}}$ (which we will also call I_C), and that it exchanges p and ∞.

(b) Show that I_C fixes each point of C, and no other points.

(c) Prove that $f(z) = 1/\bar{z}$ is inversion in the unit circle $C = C_1(0)$ by checking that it matches the description of I_C in this case.

(d) Suppose $g(z) = z - b$ for some complex number b. Show that $g^{-1} \circ f \circ g$ is inversion in the circle $C = C_1(b)$ of radius 1 with center b.

(e) Now let $h(z) = \frac{1}{R} z$ for some real number $R > 0$ and show that $h^{-1} \circ f \circ h$ is inversion in the circle $C = C_R(0)$.

(f) Using the previous parts, give a formula for inversion in the arbitrary circle $C_r(p)$.

(8) Let $f(z) = e^z - \bar{z}$. Now let L be the contour that travels the straight line from 0 to πi at unit speed.

(a) What is $f(L)$?

(b) Give the definition of derivative. At what points does $f'(z)$ exist? At what points is $f(z)$ analytic?

(c) Suppose γ is some other contour between 0 and πi. What can you say about $\int_\gamma f$?

(9) Series.

(a) Give a Taylor series for $z^2 \sin(z^2)$ about $z_0 = 0$.

(b) Give two different Laurent series for $\frac{1}{z^2 + z^3}$ about $z_0 = 0$.

(c) Suppose that $f(z)$ has a zero of order 3 at z_0 and $g(z)$ has a pole of order 2 at z_0. What can you say about $f(z)/g(z)$?

(d) What is the radius of convergence for the Taylor series of $\log(z^4)$ about $z_0 = i + 1$? (With the standard branch of log.)

(10) Let f be an entire function and suppose $a, b \in \mathbb{C}$ are any two distinct complex numbers. Let $C_R = \{Re^{i\theta}\}$ be the circle of radius R centered at 0. Let $A_R = \displaystyle\int_{C_R} \frac{f(z)}{(z-a)(z-b)}\, dz$.

(a) Evaluate the integral exactly to get an expression for A_R in terms of only f, a, and b. (You should get a few different cases depending on the size of R. A picture will help.)

(b) Supposing f is bounded, derive an upper bound for $|A_R|$ and prove that $\lim_{R \to \infty} A_R = 0$.

(c) Use the two previous parts to give a proof of Liouville's theorem!

NOTE: THE ACTUAL EXAM WILL BE AT MOST HALF THIS LONG.