1.3.24 *What are the images under vertical and horizontal lines of $z \mapsto \cos z$?*
Well,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos(-x) + i \sin(-x))}{2}.$$

Since cos is an even function and sin is an odd function, this becomes

$$\frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2} = \cos x \left(\frac{e^y + e^{-y}}{2} \right) - i \cdot \sin x \left(\frac{e^y - e^{-y}}{2} \right).$$

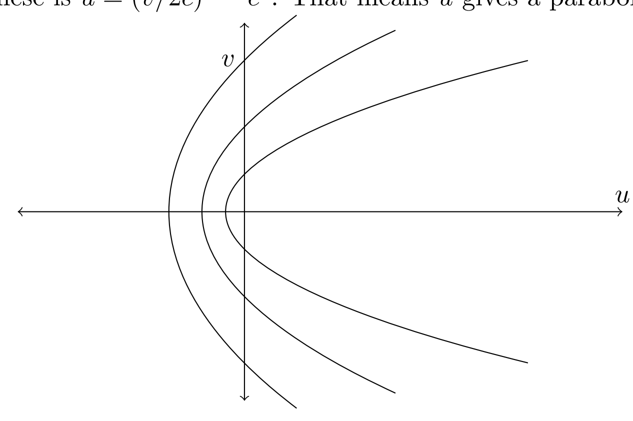
So we've shown that $\cos z = (\cos x \cosh y) - i(\sin x \sinh y)$, or in other words if $f = u + iv$ as usual then we have $u = \cos x \cosh y$ and $v = -\sin x \sinh y$.

To see where horizontal lines go, we consider fixing $y = c$ while x varies. This gives $u = a \cos x$, $v = b \sin x$, which traces out an ellipse: $(u/a)^2 + (v/b)^2 = 1$. (It's a circle that's been stretched by a horizontally and by b vertically.)

To see where vertical lines go, we consider fixing $x = c$ while y varies. This gives $u = a \cosh y$, $v = b \sinh y$. Now the most basic hyperbolic trig identity is $\cosh^2 y - \sinh^2 y = 1$, which is easy to discover by calculation or by Wikipedia. Thus we have $(v/b)^2 - (u/a)^2 = 1$, which is the equation of a hyperbola.

1.3.26 (a) *Show that under the map $z \mapsto z^2$, lines parallel to the real axis are mapped to parabolas.*

Here, we have easy formulas: since $(x + iy)^2 = (x^2 - y^2) + (2xy)i$, we have $u = x^2 - y^2$, $v = 2xy$. Lines parallel to the real axis are given by fixing $y = c$ and letting x vary. Thus we get $u = x^2 - c^2$, $v = 2xc$, and the relationship between these is $u = (v/2c)^2 - c^2$. That means u gives a parabola in v, shifted by c^2.



(b) *Show that under (a branch of) $z \mapsto \sqrt{z}$, lines parallel to the real axis are mapped to hyperbolas.*

The square root map is just the squaring map performed backwards, so let's stick with our expressions from the previous part, taking u, v back to x, y. Now lines parallel to the real axis have $v = c$ as u varies. From $v = 2xy$, we get $2xy = c$, or in other words $y = (\frac{c}{2})\frac{1}{x}$. These are hyperbolas (sketch the usual graph $y = 1/x$ to see this) with the x and y axes as their asymptotes.

Topology worksheet #1 *Show that* $\{z \in \mathbb{C} : \ |\operatorname{Re} z - a| < \epsilon_1\} \cap \{z \in \mathbb{C} : |\operatorname{Im} z - b| < \epsilon_2\}$ *is an open rectangle in the complex plane. Using this, show that for every disk in the plane, there are real numbers* $a, b, \epsilon_1, \epsilon_2$ *such that the rectangle they describe lies inside the given disk.*

Let the intersection of those sets be called R. It is the locus of points whose x values are between two values and whose y values are between two values; the intersection of those two strips is a rectangle.

I need to show R is open. I will show that for any point in R, it has an open disk neighborhood totally within R. Take any $p = x + yi \in R$. By the definition of R, this means that $a - \epsilon_1 < x < a + \epsilon_1$ and $b - \epsilon_2 < y < b + \epsilon_2$.

Let $\delta = \min\{(a + \epsilon_1) - x, x - (a - \epsilon_1), (b + \epsilon_2) - y, y - (b - \epsilon_2)\}$, which is the smallest vertical/horizontal distance from p to any of the sides of the rectangle. Then the disk $D(p; \delta/2)$ is totally contained in R.

Now let $D(q; r)$ be an arbitrary disk in the plane. For the rectangle R described in this problem, the farthest point from the center is $\sqrt{\epsilon_1^2 + \epsilon_2^2}$, so we need the rectangle to be centered at q and we need $\sqrt{\epsilon_1^2 + \epsilon_2^2} < r$. So it will certainly work if we take $a = \operatorname{Re}(q)$, $b = \operatorname{Im}(q)$, and $\epsilon_1 = \epsilon_2 = r/2$.

1.5.14 (e) *Show that the expression* $\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} z^n \bar{z}^m$ *is an analytic function of* z *if and only if* $a_{nm} = 0$ *whenever* $m \neq 0$.

In earlier parts of this problem, you showed that $\partial/\partial z$ and $\partial/\partial \bar{z}$ are linear operators satisfying the product rule; the first gives 1 and 0 when applied to z and $\bar{z}$, respectively, while the second does the opposite. You also showed that a function f is analytic if and only if $\partial f/\partial \bar{z} = 0$.

The problem is asking us to show that the expression can only be analytic if there are no (nonzero) terms with $m > 0$. So to finish this problem, we only need to show that if $m > 0$, then the expression is NOT analytic; that is, it is not killed by $\partial/\partial \bar{z}$.

Now of course z^n is analytic, so ${}^{\partial}/_{\partial \bar{z}}(z^n) = 0$. On the other hand, ${}^{\partial}/_{\partial \bar{z}}(\bar{z}^m) = m\bar{z}^{m-1}$, as you can check with a computation or just deduce because $\partial/\partial \bar{z}$ works on $\bar{z}$ just the way the usual derivative works on z.

$$\frac{\partial}{\partial \bar{z}} \left(\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} z^n \bar{z}^m \right) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} \frac{\partial}{\partial \bar{z}} \left(z^n \bar{z}^m \right)$$

So let's just consider the individual $\frac{\partial}{\partial \bar{z}}(z^n \bar{z}^m)$ terms:

$$\frac{\partial}{\partial \bar{z}}(z^n \bar{z}^m) = z^n \frac{\partial}{\partial \bar{z}}(\bar{z}^m) + \bar{z}^m \frac{\partial}{\partial \bar{z}}(z^n) = m z^n \bar{z}^{m-1},$$

which is a nonzero function as long as $m > 0$. And no other term in the original sum has the same combination of z and $\bar{z}$ terms to cancel it out.

So we've shown what we wanted: if $m > 0$, then the $\partial/\partial \bar{z}$ operator does not send our expression to the zero function, so it's not analytic.

1.5.18 *Let* $f(z) = z^5/|z|^4$ *if* $z \neq 0$ *and* 0 *if* $z = 0$.

(a) *Show that* $f(z)/z$ *does not have a limit as* $z \to 0$.

Note that if $z = re^{i\theta}$, then $|z| = r$, so $f(z) = re^{5i\theta}$. Thus $f(z)/z = e^{4i\theta}$, which is the point on the unit circle with argument 4θ. Thus $f(z)/z$ takes different values as we approach from different angles, so it has no overall limit.

(b) *If* $u = \mathrm{Re}(f)$ *and* $v = \mathrm{Im}(f)$, *show that* $u(x,0) = x$, $v(0,y) = y$, $u(0,y) = v(x,0) = 0$.

From our formula above, we see that $u = r\cos(5\theta)$ and $v = r\sin(5\theta)$. Now $(x,0)$ is a point on the x axis, so it has $\theta = k\pi$, for some whole number k, and since $5\theta = 5k\pi = k\pi + 4k\pi$, we see that $5k\pi$ and $k\pi$ have the same argument. Likewise $(0,y)$ has argument $\theta = \pi/2 + k\pi$, and we have $5\theta = 5\pi/2 + 5k\pi = \pi/2 + k\pi + 2\pi + 4k\pi$, which differs by a multiple of 2π again. That means $u = r\cos(5\theta) = r\cos(\theta) = x$ and $v = r\sin(5\theta) = r\sin(\theta) = y$ for all points on either of the two axes. Thus $u(x,0) = x$, $u(0,y) = 0$, $v(x,0) = 0$, and $v(0,y) = y$.

(c) *Conclude that the partials of* u, v *exist and that the Cauchy-Riemann equations hold but that* $f'(0)$ *does not exist. Does this conclusion contradict the Cauchy-Riemann theorem?*

Why do the partials u_x, u_y, v_x, v_y exist? Because $u = x$ and $v = y$ along the x-axis, and the same thing is true along the y-axis, so the directional derivatives are the same as they would have been for the identity function:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Why do the CR equations hold? Because that matrix is in the right form, $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$.

Why doesn't $f'(0)$ exist? Well, consider the definition of derivative!

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

But considering that $f(0)$ was defined to be zero, this can be rewritten as $f'(0) = \lim\limits_{z\to 0} \frac{f(z)}{z}$. But that's the limit that we showed in part (a) does not exist!

Now, of course this doesn't contradict the CR theorem, because after all it's a theorem. The loophole is that for the matrix of partial derivatives to be a Jacobian at all (in other words, for the function to be differentiable even in the sense of real variables), the partials have to be continuous, and here we never verified that. This whole problem is actually designed to show you why that extra little hypothesis is necessary.

1.5.20 *Let* f *be an analytic function on an open connected set* A *and suppose that* $f^{(n+1)}(z)$ *(the* $(n+1)$*st derivative) exists and is zero on* A. *Show that* f *is a polynomial of degree* $\leq n$.

OK, first let's establish this for $n = 0$. That is: if the first derivative of a function is zero on A, then the function is polynomial of degree zero. This was shown in class: if $f'(z) \equiv 0$ on a domain A, then $f(z) \equiv c$ on A.

From this we can establish a Claim: *any two functions with the same derivative on a domain* A *must differ only by a constant on* A.

Proof: Suppose g and h satisfy $g'(z) = h'(z)$ on A. Then let $f = g - h$. We have $f'(z) = g'(z) - h'(z) = 0$ on A, so by the previously cited fact, $f(z) \equiv c$ on A. Thus we've shown $g(z) = h(z) + c$ on A, as desired. $\square$

So now let's try this for $n = 1$. We suppose that $f''(z) = 0$. This is the derivative of f', so by the fact from class, we have $f'(z) = c$. So what is f itself? Well, I know one function whose derivative is c, namely the function $F(z) = cz$. So my function f must only differ from it by a constant, giving $f(z) = cz + d$, which is a polynomial of degree at most one.

I can continue to increment n one step at a time (or construct a proof by induction if you know what that is). Here's the end using proof by induction: Suppose the claim is established for all $n < N$ and let's prove it for $n = N$. Now $f^{(N+1)}(z) \equiv 0$, which is the Nth derivative of f', so f' is a polynomial of degree at most $N-1$. This has one antiderivative that we know, which is a polynomial of degree N. So f can only differ from that by a constant, making it also a polynomial of degree N.

ALTERNATE ARGUMENT: $f^{(N+1)}$ is the first derivative of $f^{(N)}$, so if that is zero it follows that $f^{(N)}$ is constant, say c. Now I know a function, namely $H(z) = \frac{c}{N!}z^N$ (a polynomial of degree N), whose Nth derivative is c. So I have $f^{(N)}(z) - H^{(N)}(z) = 0$, so the Nth derivative of $f - H$ is zero, so by inductive hypothesis, $f - H$ is a polynomial of degree at most $N-1$. That means f is the sum of H and that polynomial, so f is a polynomial of degree at most N.

2.2.8 *Let γ_1 be the circle of radius 1 and let γ_2 be the circle of radius 2 (traversed counterclockwise and centered at the origin). Show that*

$$\int_{\gamma_1} \frac{dz}{z^3(z^2+10)} = \int_{\gamma_2} \frac{dz}{z^3(z^2+10)}$$

This problem will be done if we can apply the Deformation Theorem to transform one curve to the other by a homotopy. In order to do that, we need to check that the region between the two curves (which is the region slid over by the homotopy) is a domain of analyticity for the function f that we are integrating. Since the integrand f is a rational function, it is analytic everywhere that it is defined. So only the zeroes of the denominator are singularities for the function. These occur at $z = 0$ and at the square roots of -10, which are $\pm\sqrt{10}\cdot i$. Since $\sqrt{10} > 2$, none of these singularities falls in the ring between γ_1 and γ_2.

2.2.10 *Evaluate $\int_\gamma \sqrt{z^2-1}\, dz$, where γ is a circle of radius $\frac{1}{2}$ centered at 0.*

In Worked Example 1.6.8 (page 88) it is explained how to work with a nonstandard branch of the square root function, slit along the POSITIVE real axis instead of the usual negative real axis, such that the domain of analyticity for $\sqrt{z^2-1}$ becomes $\mathbb{C}\setminus\Big((\infty,-1]\cup[1,\infty)\Big)$. That domain is simply connected. For this branch, the whole contour γ is inside this simply connected domain, so by Cauchy's Theorem, the integral must be zero.

HOWEVER, when you change the branch of the function you're integrating, you may have changed the answer! So if the problem is assuming working with the standard branch for the square root function, this is not good enough.

There is a way to integrate right through branch cuts using limits: suppose you want to compute $\int_\gamma f$ for a curve parametrized over $[a,b]$ but $\gamma(c)$ is on a branch cut. Then you can just let γ_1 be the part of γ defined on $[a, c-\epsilon]$ and let γ_2 be the part of γ defined on $[c+\epsilon, b]$, so that if all the limits exist, then

$$\int_\gamma f = \lim_{\epsilon\to 0}\left[\int_{\gamma_1} f + \int_{\gamma_2} f\right] = \lim_{\epsilon\to 0}\left[\int_a^{c-\epsilon} f(\gamma(t))\cdot\gamma'(t)dt + \int_{c+\epsilon}^b f(\gamma(t))\cdot\gamma'(t)dt\right].$$

So in theory, that's another way to do this problem without changing to a nonstandard branch... but in practice, I tried it and it works out to be horrible!