

2.5.18 *The function $f(z)$ is analytic over the whole complex plane and $\operatorname{Im} f \leq 0$. Prove that f is constant.*

We write $w = f(z) = u + iv$, as usual, so that u and v are real-valued functions of z , with $v = \operatorname{Im} f$, which means $v \leq 0$ for all z . Then note that since $f : z \mapsto w$ is an analytic function, the function $z \mapsto e^{-iw} = e^{v-iu}$ is analytic as well (since it are obtained by composing f with multiplication by $-i$ and then by exponentiation, which are each analytic). The norm of $e^z = e^{x+iy}$ is just e^x , and the argument is y , so the norm of e^{-iw} is e^v . Since $v \leq 0$, we have $|e^v| \leq 1$, so this is a bounded, entire function. By Liouville's theorem, it is constant. If e^{-iw} is constant, then $-iw$ must be constant as well, so w is constant, as desired.

5.2.18 *Suppose that a, b, c, d are all real and that $ad > bc$; show that $T(z) = \frac{az+b}{cz+d}$ leaves the upper half-plane invariant. Show that every conformal map of the upper half-plane onto itself is of this form.*

We want to show that $T(\mathbb{H}) = \mathbb{H}$. Let $w = T(z)$; we must show that if $\operatorname{Im} z > 0$, then $\operatorname{Im} w > 0$. We have

$$w = \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2},$$

whose numerator is $ac|z|^2 + adz + bc\bar{z} + bd$ and whose denominator is positive. (It can't be zero because $-d/c$ is on the real line or infinity, not in \mathbb{H} .) The only contribution to the imaginary part is $adz + bc\bar{z}$ (it will be divided by something positive, so that won't change its sign). If $z = x + iy$, then the imaginary part of that is $ady - bcy = (ad - bc)y$. We've assumed $y > 0$, and the problem tells us that $ad > bc \implies ad - bc > 0$, so the imaginary part of w is indeed positive too.

For the second part, we know (Prop 5.2.2 in the book) that all conformal self-maps of the unit disk \mathbb{D} are Möbius transformations; we need to use that to show the same for all conformal self-maps of \mathbb{H} . Suppose $f : \mathbb{H} \rightarrow \mathbb{H}$ is conformal. Let T be a Möbius transformation that takes \mathbb{H} to \mathbb{D} , such as $T(z) = \frac{z-i}{z+i}$. Note T^{-1} is also a Möbius transformation, and indeed with a formula that is easy to write down if we need to. Then $g = T \circ f \circ T^{-1}$ is a map from $\mathbb{D} \rightarrow \mathbb{D}$, and it is conformal because it is the composition of three conformal maps, so it is Möbius by the proposition. Then we have $f = T^{-1} \circ g \circ T$ is the composition of three Möbius transformations, so it is itself Möbius, as desired.

5.2.26 *Conformally map $A = \{z : |z - 1| < 1\}$ onto $B = \{z : \operatorname{Re} z > 1\}$.*

The first set is points within 1 unit of 1, so it is an open disk $D(1; 1)$. The second set is points above the horizontal line $y = 1$. First take $f(z) = z - 1$, so that $f(A)$ is the standard unit disk centered at the origin. Then take $g(z) = \frac{iz+i}{-z+1}$, which sends \mathbb{D} to \mathbb{H} , and was obtained as the inverse of the map from the last problem. Finally, let $h(z) = z + i$, which translates up by one. Then $h \circ g \circ f$ sends A to B .

5.3.3 *Find the electric potential in the region illustrated in the figure.*

The problem asks us to find a harmonic function on a vertical half-strip meeting certain boundary conditions: we want $\phi = 1$ on the left and right sides and $\phi = 0$ on the bottom. To do this, we should find a conformal map sending the region to

the upper half-plane, then use the standard solution to the Dirichlet problem on \mathbb{H} , then take the composition of the two to solve the problem.

Figure 5.2.11 in the book (p341) shows an assortment of conformal maps between various regions of the plane. We see in (xi) that $f(z) = \sin z$ sends precisely our vertical half-strip to the upper half-plane, sending the vertices to $x = \pm 1$.

The standard upper half-plane solution, if $x_1 = -1$, $x_2 = 1$, $c_0 = 1$, $c_1 = 0$, $c_2 = 1$, is

$$u(x, y) = 1 + \frac{1}{\pi} [(0 - 1)\theta_2 + (1 - 0)\theta_1] = 1 - \frac{1}{\pi} \left(\arctan \frac{y}{x - 1} \right) + \frac{1}{\pi} \left(\arctan \frac{y}{x + 1} \right),$$

where θ_i is the angle made between the positive x -axis and the point (x, y) at x_i . Those arctan formulas are obtained by forming a right triangle between (x, y) , ± 1 , and $(x, 0)$. The vertical side has length y and the horizontal side has length $x \pm 1$, so the angle is the arctan of the ratio.

So to get a map on the original region, we first do f , then u , so $u \circ f$ is the map we seek. Splitting up \sin into its real and imaginary parts, we have $\sin(x + iy) = \sin x \cosh y + i \sinh y \cos x$, so these become the x and y in our formula, and finally we have

$$u_1(x, y) = 1 - \frac{1}{\pi} \arctan \frac{\sinh y \cos x}{\sin x \cosh y - 1} + \frac{1}{\pi} \arctan \frac{\sinh y \cos x}{\sin x \cosh y + 1}.$$

5.3.6 Suppose a point with charge $+1$ is located at $z_0 = (1 + i)/\sqrt{2}$ and the positive real and imaginary axes are a grounded conductor maintained at potential zero. Find the potential at every point $z \neq z_0$ inside the region A .

Example 5.3.4 explains how to find a standard solution for electric potential in the presence of a point charge: put $\phi(z) = \frac{Q}{2\pi} \log |z - z_0| + K$, where Q is the charge at z_0 and K is an appropriate constant. Since this formula only depends on $|z - z_0|$, or the distance of z from z_0 , it is clear that the equipotential curves are circles centered at z_0 . For our application, we want the axes to be equipotential curves, so we need to apply a conformal map taking the first quadrant to a circle. We know how to take \mathbb{H} to \mathbb{D} , so let's first apply $f(z) = z^2$ to our region, taking the first quadrant to \mathbb{H} . Then we'll use $g(z) = \frac{z-i}{z+i}$ to get to the disk. The composition is $g \circ f(z) = \frac{z^2 - i}{z^2 + i}$. If we are lucky, z_0 will have been mapped to the center. We can compute that $z_0^2 = i$, so indeed $g(f(z_0)) = 0$; we are in luck. On the unit disk, the standard solution for charge $+1$ is $\phi(z) = \frac{1}{2\pi} \log |z|$, and we can confirm that this function has $\phi = 0$ on the boundary of \mathbb{D} . So our final answer will be

$$\phi_1(z) = \frac{1}{2\pi} \log \left| \frac{z^2 - i}{z^2 + i} \right|.$$