Real Analysis

a short presentation on what and why

I. Fourier Analysis

Fourier analysis is about taking functions and realizing them or approximating them in terms of periodic (trig) functions. Many, many practical applications.

Def: $\mathcal{R}$ = { Riemann-integrable functions $[-\pi,\pi] \longrightarrow \mathbb{R}$ }

$\ell^2(\mathbb{Z})$ ={ infinituples s.t. $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$}; we have a nice map $\mathcal{R} \longrightarrow \ell^2(\mathbb{Z})$ via

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

Fourier, continued

It's nice because of Parseval's identity, which guarantees that

$$\| (a_n) \|_2 = \sum_{-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \| f \|_2$$

i.e., the correspondence is norm-preserving. BUT the map $\mathcal{R} \longrightarrow \ell^2(\mathbb{Z})$ is not surjective, and only the target is complete (contains limits of Cauchy seqs).

That means there are tuples in $\ell^2(\mathbb{Z})$ that don't correspond to Riemann-integrable functions.

What ARE the functions with these coefficients? How do we integrate them?

II. Limits

Recall that C[0,1] denotes continuous functions $[0,1]\longrightarrow\mathbb{R}$. Given a sequence of such functions, suppose the limit exists pointwise:

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x$$

What can we say about *f* ? Recall def. of *uniform convergence*:

$$\forall \varepsilon > 0 \;\; \exists N \;\; s.t. \;\; |f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \forall x$$

If $f_n \longrightarrow f$ uniformly, then *f* is continuous. Otherwise, it's not necessarily even Riemann-integrable. Can we find a way to integrate that is "nice," i.e.,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \quad ?$$

III. Length

For a curve γ(t)=(x(t), y(t)), it's called *rectifiable* if the two definitions of length agree:

$$\int_a^b \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\ dt \quad = \quad \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})),$$

where the sup is over subdivisions $a = t_0 < t_1 < \ \cdots < t_n = b$

When (for what conditions on the coordinate functions *x*, *y*) does the sup converge? Does that force the integral to exist? What are the geometric properties of rectifiable curves, and what are non-rectifiable curves like?

IV. Fundamental Theorem of Calculus

Remember the FTC? It's got two forms,

$$F(b) - F(a) = \int_a^b F'(x)\,dx \quad \text{and} \quad \frac{d}{dx}\int_0^x f(y)\,dy = f(x).$$

For which *F* does the FTC hold?

What's the full generality of differentiating the integral?

V. Measure

All of these mysteries cluster around questions of measure and singularity. Measures are "things we can integrate against" (like our enigmatic *dx*) and they tell us "how much" in maximal generality.

Def: For a set X, we want a collection $\Sigma \subseteq \mathcal{P}(X)$ of "measurables" and

$\mu : \Sigma \longrightarrow [0,\infty]$ s.t. $\mu(\varnothing)=0$ and μ is *countably additive*:

$$\mu\Big(\coprod_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i)$$

It would be particularly lovely to have one on $\mathbb{R}^d$ that is translation-invariant and "nice" on intervals:

$$\mu\Big(\prod\, [a_i, b_i]\,\Big) = \prod (b_i - a_i) \;\; ; \quad \mu(E) = \mu(E + v)$$

Measures, intuition, and paradoxes

Lebesgue measure is the gadget we seek on $\mathbb{R}^d$, but it has some crazy-seeming properties and it does not "play well" with certain intuitively benign principles like the Axiom of Choice.

First the good news: besides satisfying our requirements, Lebesgue measure gives an integral that is quite strong and flexible. It can deal with "horrible" functions like the classic $\chi_{\mathbb{R}\text{-}\mathbb{Q}}$. The integral of $\chi_{\mathbb{R}\text{-}\mathbb{Q}}$ over [0,1] is 1, because Lebesgue measure "knows" that there are way more irrationals than rationals.

As a function *m*: $\mathbb{R}^d \longrightarrow [0,\infty]$, it should be thought of as giving *length* (d=1), *area* (d=2), or *volume* ($d\geq3$).

That was the good news; on the other hand, if you accept AC, then there are non-measurable sets in $\mathbb{R}$, such as Vitali sets.

AC: For every collection of sets, it is possible to choose one representative from each.

Def: a *Vitali set* is a set of representatives for the equivalence relation

$$x \sim y \iff x - y \in \mathbb{Q}$$

Philosophical note: should you accept AC? It's kind of up to you, since it was proved independent of ZF, the usual axioms of set theory. Most people accept it and work in ZFC. You can always be a holdout and work in ZFS, which is ZF plus the axiom that *all sets are Lebesgue-measurable*!

Paradoxes and riddles

Banach-Tarski Paradox (1924): It is possible to take a solid unit ball in $\mathbb{R}^3$, divide it into five pieces, then rotate the pieces to reassemble two solid unit balls.

This doubles volume, of course, and rotations are rigid motions, which should preserve geometric quantities like volume, so that seems bad.

Answer: B-T is possible if you accept AC, in which case it relies on a decomposition into non-measurable pieces.

Paradoxes and riddles

Hadwiger-Nelson Problem (1950-??): How many colors does it take so that every point of the Euclidean plane can be assigned a color, and no two points at distance one have the same color?

Answer: it's only known that at least 4 colors are needed and that 7 suffice. HOWEVER, if it's possible with 4 or 5, the color-sets must be non-measurable.

Paradoxes and riddles

Ruziewicz Problem (~1920): what are the rotation-invariant measures on the sphere S^d that are defined on all Lebesgue-measurable sets? What if we relax countable additivity and only require the measure to be *finitely additive*?

Answer: *Lebesgue is it* for $d \geq 2$. For $d \geq 4$, this was proved by Margulis and Sullivan (independently) circa 1980. For $d=2,3$, Drinfeld did it in 1984. The methods include Lie groups, representation theory, number theory....

These three guys have three Fields medals.

Tiny glimpse into the history

In Newton/Leibniz calculus (17th c.), the basic objects are series, not functions. Thinking about it functionally, this amounts to a standing assumption that all functions are *analytic* (represented by a convergent power series). Most of the manipulations were done without worrying about convergence.

Abel 1826: "Analysts have been mostly occupied with functions that can be expressed as power series. As soon as other functions enter---which certainly is rarely the case---one does not get on any more.... an infinite **multitude of mistakes** will follow."

The history from the 1660s to the 1970s is a fitful exploration of the generality and foundations of calculus.

Quick & Dirty Timeline

1660s-70s: Newton and Leibniz introduce a calculus of series with rudimentary derivatives and integrals, relying on murky reasoning about infinitesimals
18th c.: Euler introduces functions, reworks "analysis infinitorum"
1790s-1800s: Fourier series and Parseval's identity
1816: Bolzano defines limits, continuity, uniform continuity (not really known till 1870s)
1820s-40s: Cauchy publishes prolifically on "rigorous" real analysis; admits there are "exceptions" (e.g., *nobody* understands L'Hôpital's rule)
1850s: Riemann gives first limit definition of integrals
1850s-80s: Weierstrass develops uniform convergence; 1872 finds cont, nowhere diff functions
1870s: several constructions of the real numbers (Dedekind cuts, Cauchy sequences, ...)
1880s-90s: Arzelà-Ascoli: equicontinuity
1880s-90s: Cantor tries to understand how bad singularities can be, blows up mathematics
1898-99: infusion of topology: Borel sets, Borel measure
1899: Baire Category Theorem: powerful tool in point-set topology
1901-02: Lebesgue measure, Lebesgue integral; 1905 Zermelo blows things up again with AC Borel: "such reasoning does not belong in mathematics"; Baire, Borel, Lebesgue jump ship
1910s-50s: Egorov, Luzin, and the Moscow School ("Luzitania") carry the torch
1910s: Hausdorff measure, Hausdorff dimension, gateway to fractals
1970s: Robinson finally gives a rigorous mathematics of infinitesimals ("non-standard analysis")

Pathology and perversion

Liouville 1842, on Cauchy's work: "It is perhaps good to add in general that rules of the kind we have just discussed and that deal with singular and exceptional values almost always have **cases of failure** that result from their very nature. One must use them with reserve and must assure oneself, in each of the examples in which one uses them, that the use is legitimate."

Hermite 1893, to Stieltjes: "I turn aside with a shudder of horror from this **lamentable plague** of functions which have no derivatives."

Pathology and perversion

Poincaré 1908: "**Logic sometimes makes monsters.** Since half a century we have seen arise a **crowd of bizarre functions** which seem to try to resemble as little as possible the honest functions that serve some purpose. No longer continuity, or perhaps continuity, but no derivatives, etc. Nay more, from the logical point of view, it is these strange functions which are the most general, those one meets without seeking no longer appear except as a particular case. There remains for them only a small corner. Heretofore when a new function was invented, it was for some practical end; to-day they are invented expressly to put at fault the reasonings of our fathers, and one never will get from them anything more than that."

So, real analysis: what and why?

This course will develop real analysis as a solid conceptual framework for calculus, introducing definitions and constructions that have been refined over 300 years to handle most of that period’s “paradoxes” and “pathologies.”

We’ll present theories of **measure**, **integration**, and **function spaces** to build up the most modern, rigorous, and general setting we have for functions of a real variable.

The techniques will bring together algebra, geometry, set theory, and topology with an emphasis on estimates, continuity, and uniformity and a strict axiomatic method that add up to **classical analysis**.