INTRODUCTION TO NILPOTENT GROUPS

Moon Duchin

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- $\leq s$-step nilpotent $\Longleftrightarrow$ $(s+1)$-fold commutators are killed

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(proved by embedding the Lie algebra into strictly upper△s)

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all three generalize!

cf: Mal'cev theory

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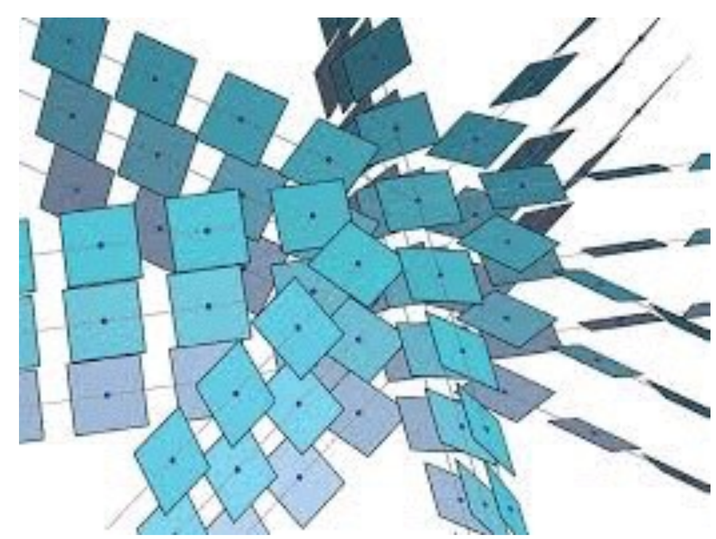

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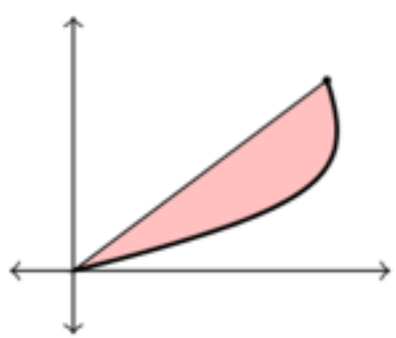

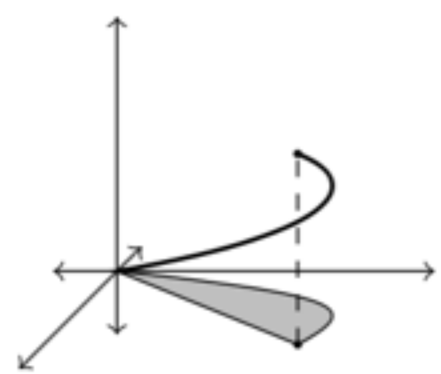

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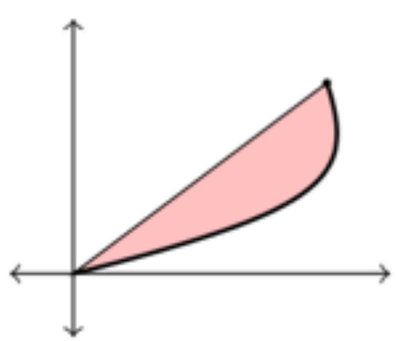

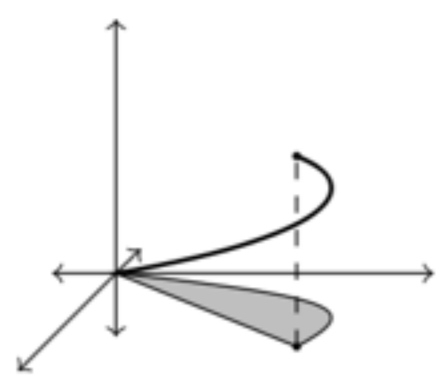

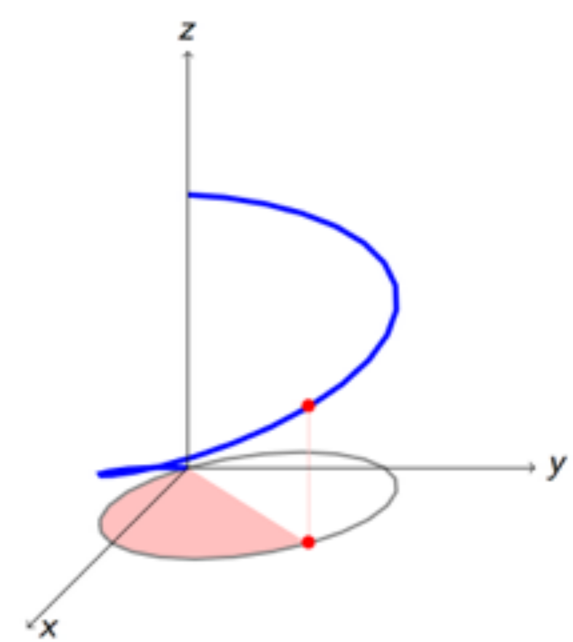

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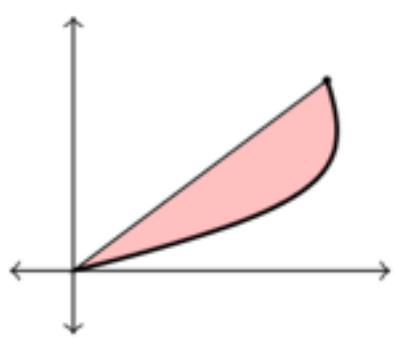

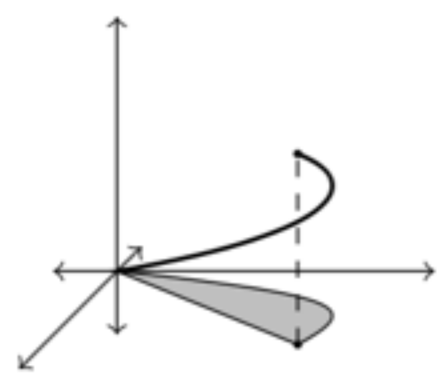

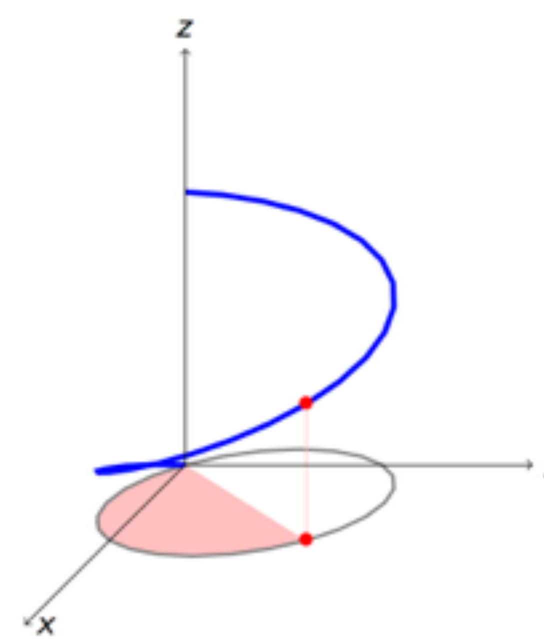

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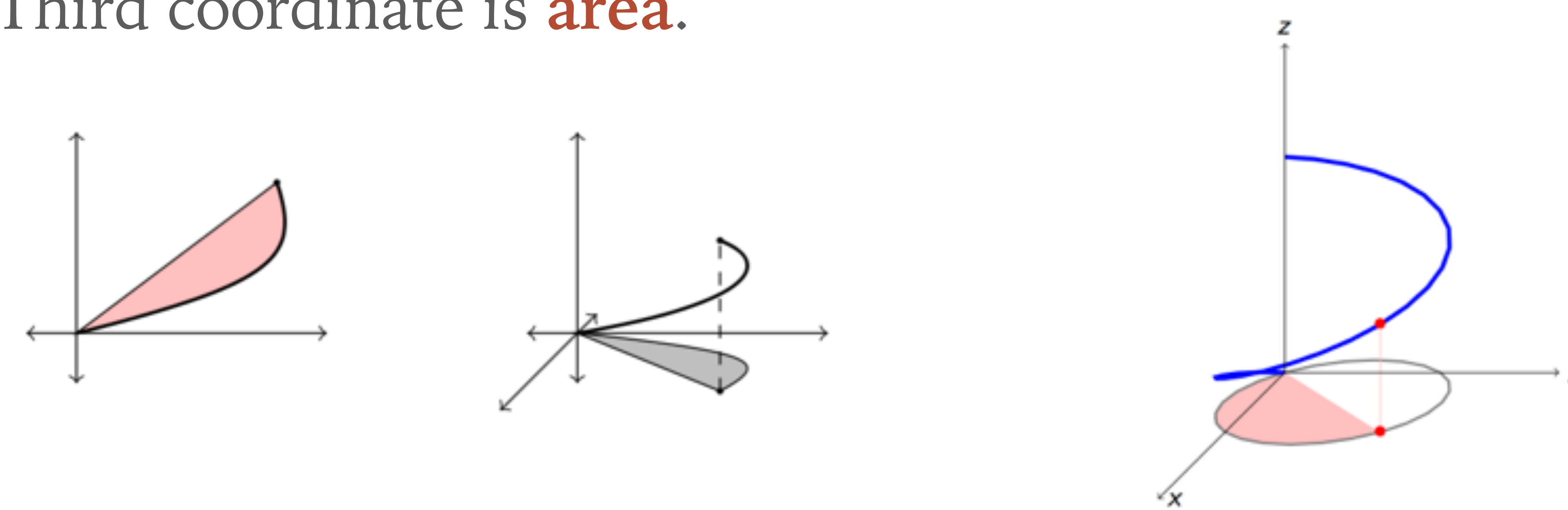

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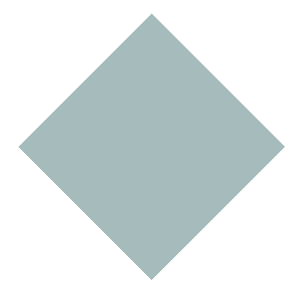

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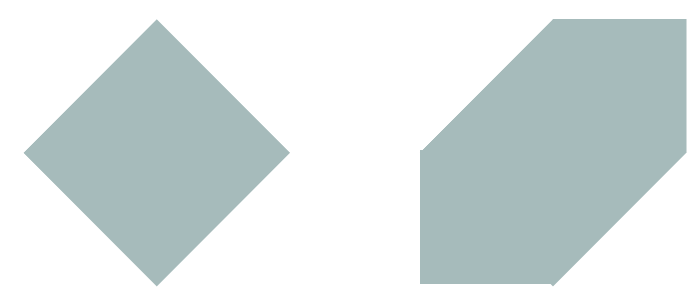

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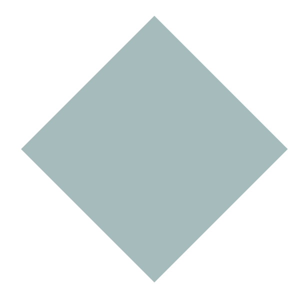


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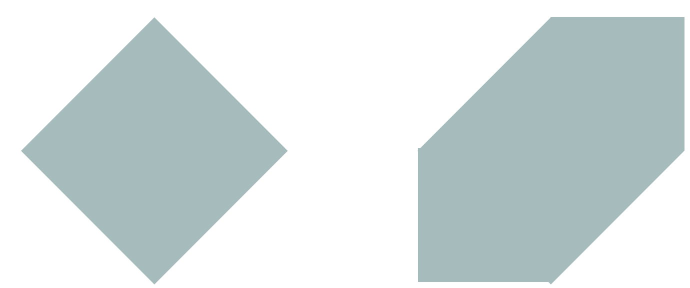

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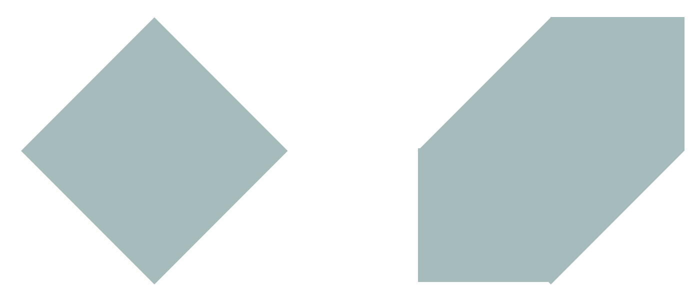

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Amazing fact: this ***characterizes*** *Carnot groups!*

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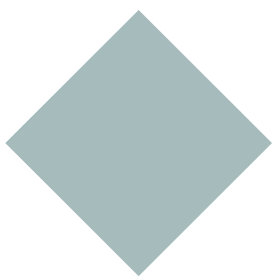

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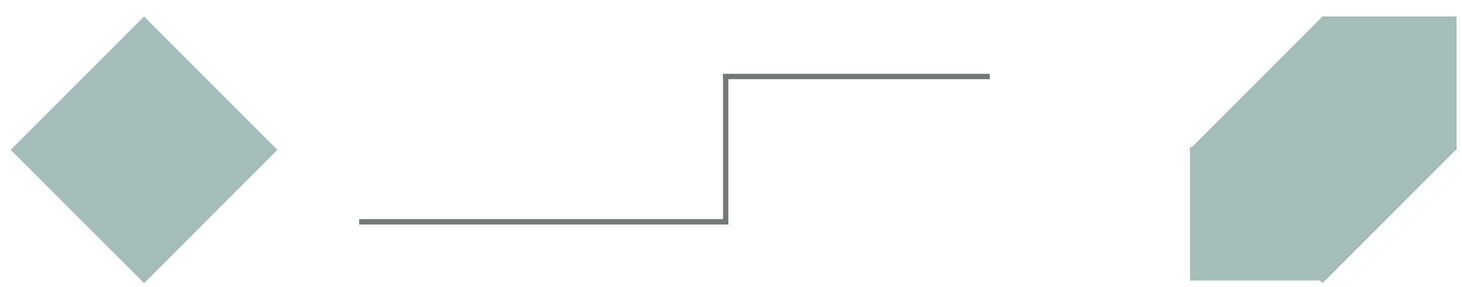

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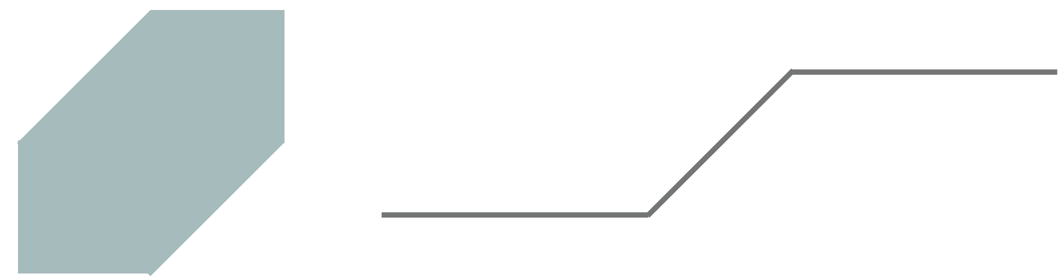

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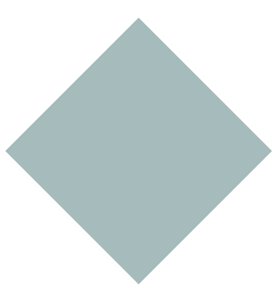

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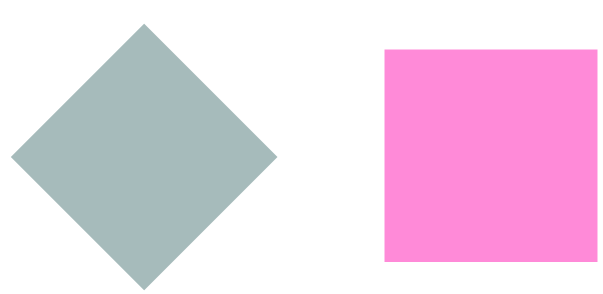

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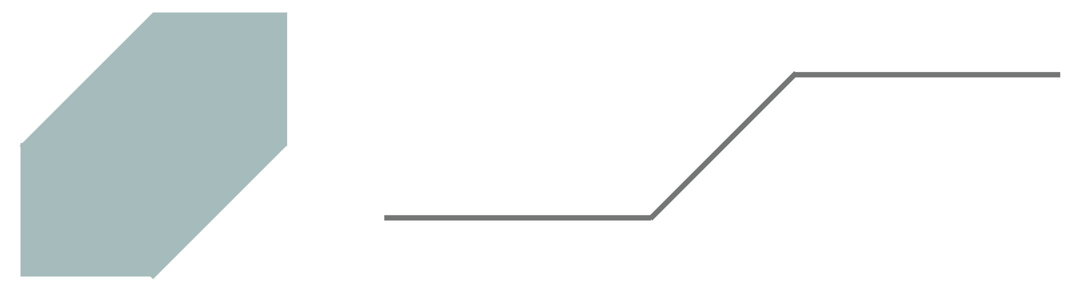

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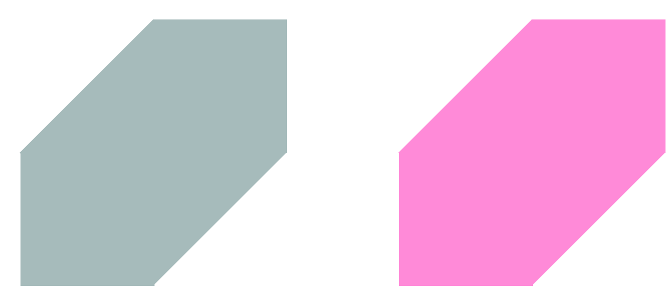

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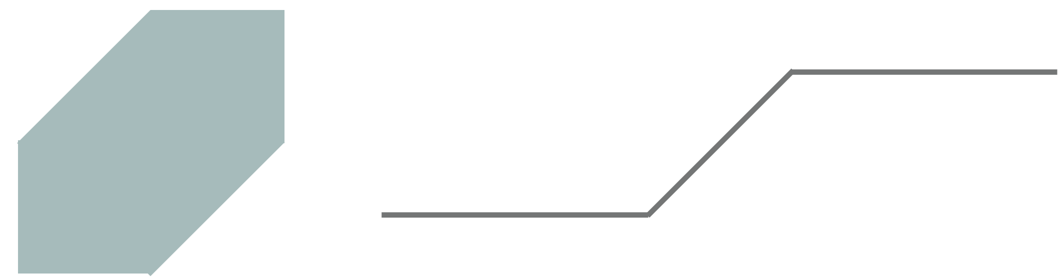

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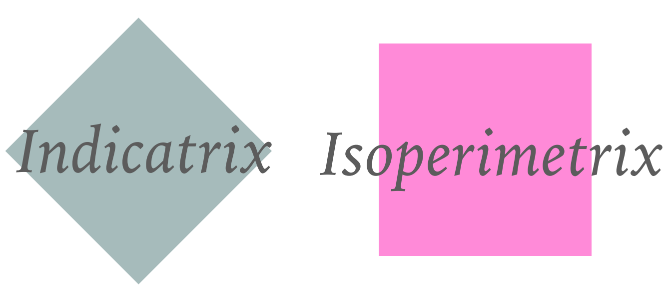

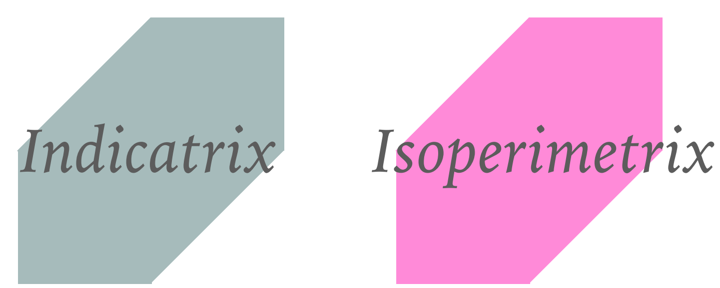

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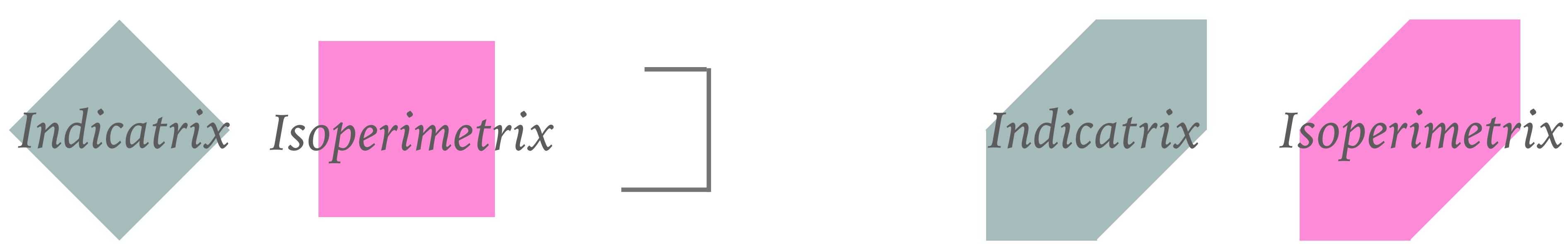

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- Let's call these "**beelines**" and "**area grabbers**."

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- L^2 case: isoperimetrix is a circle; beelines are straight horizontal lines; area-grabbers are circular spirals

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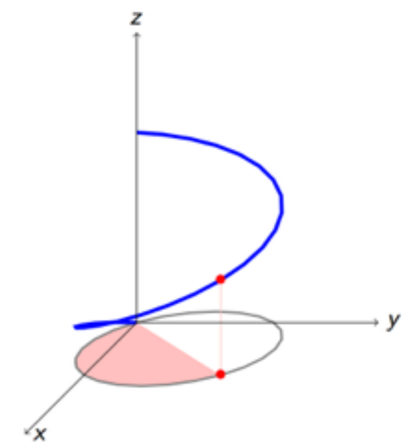

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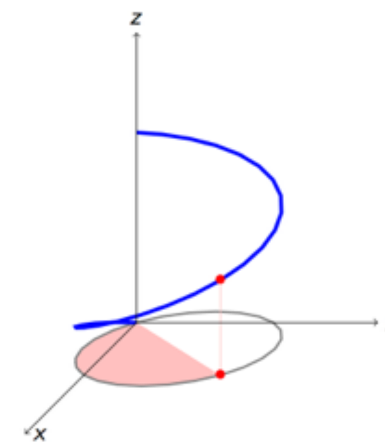

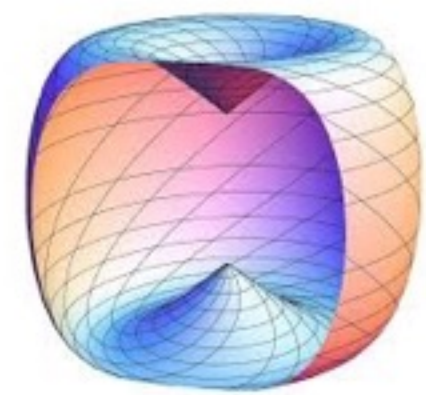

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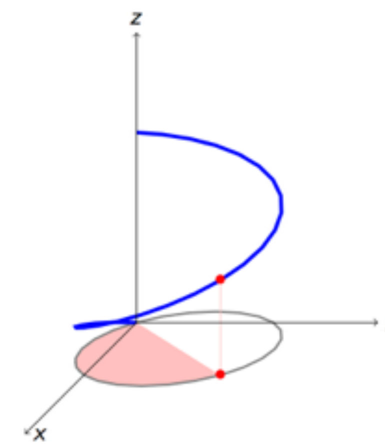

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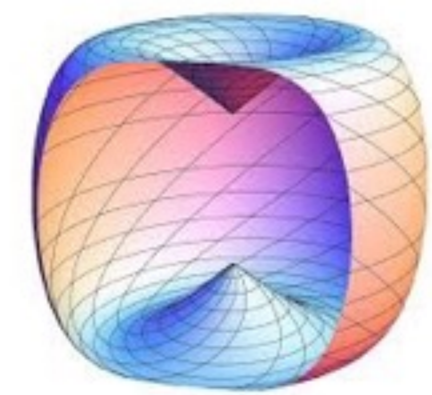

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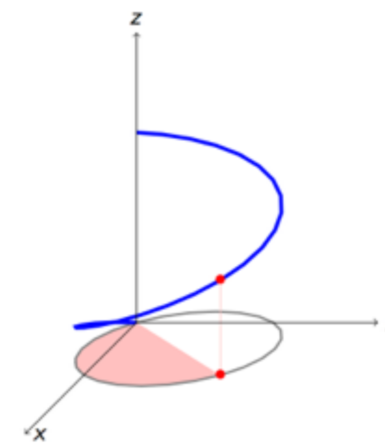

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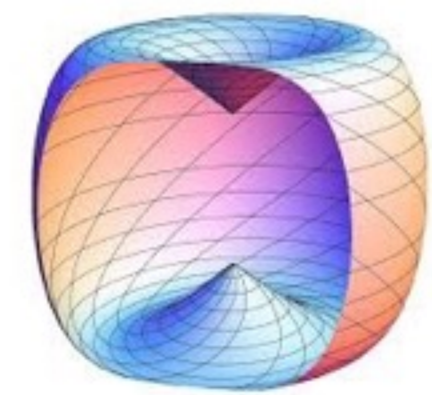

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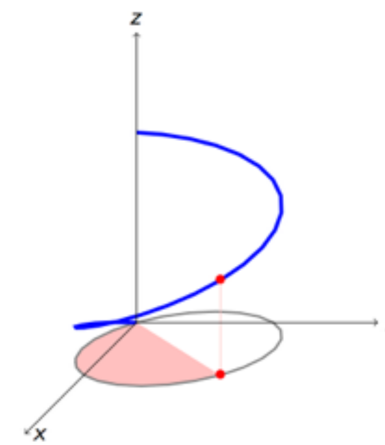

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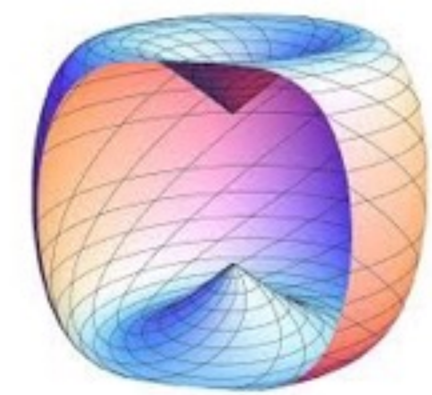

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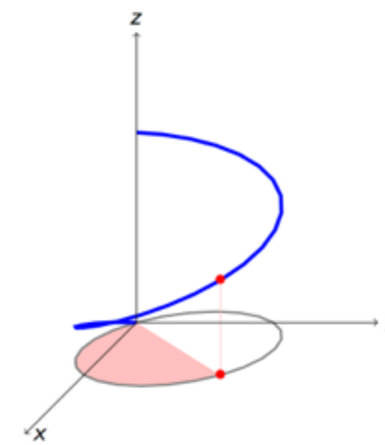

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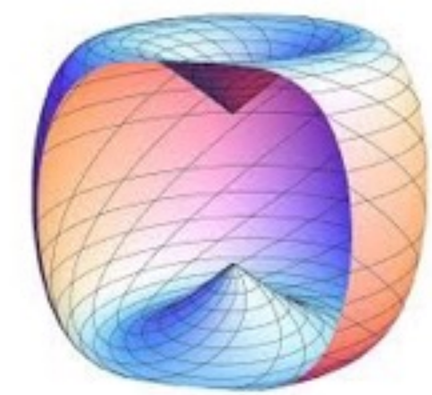

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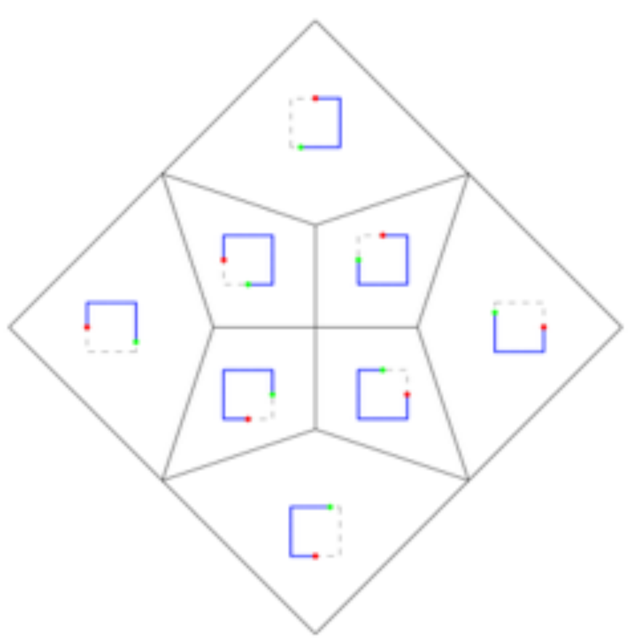

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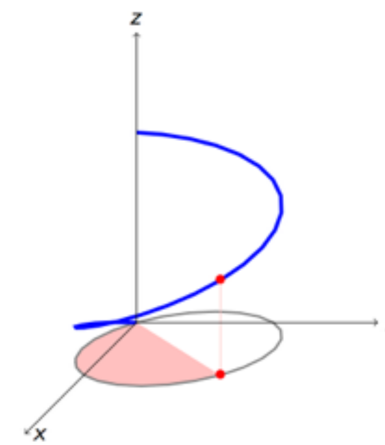

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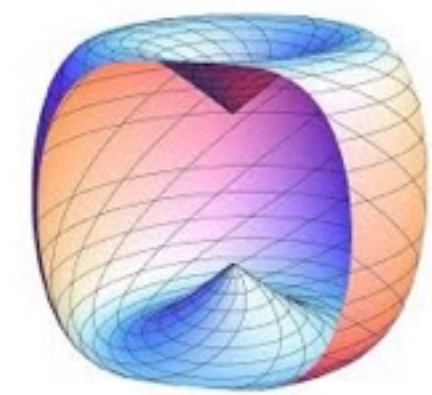

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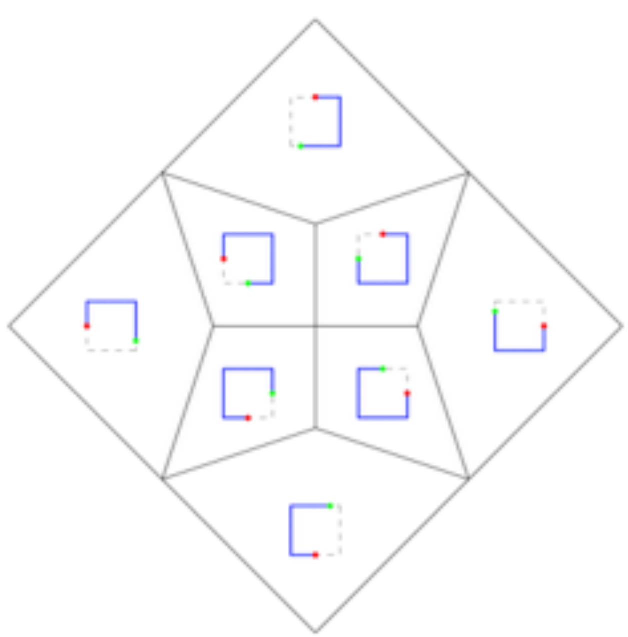

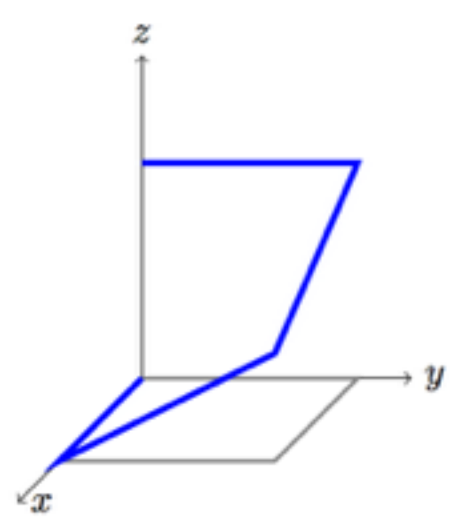

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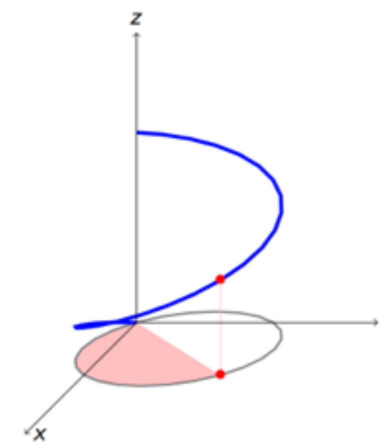

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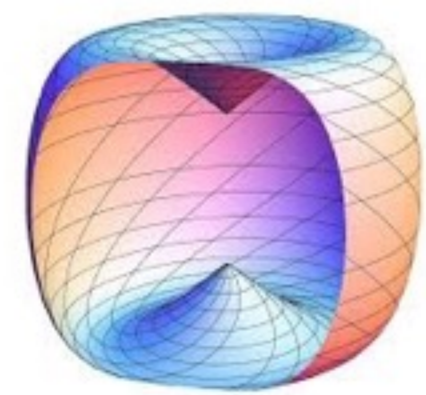

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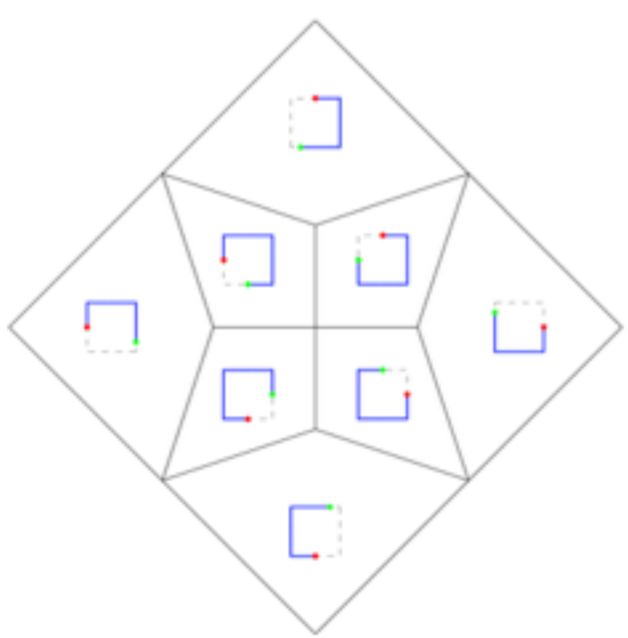

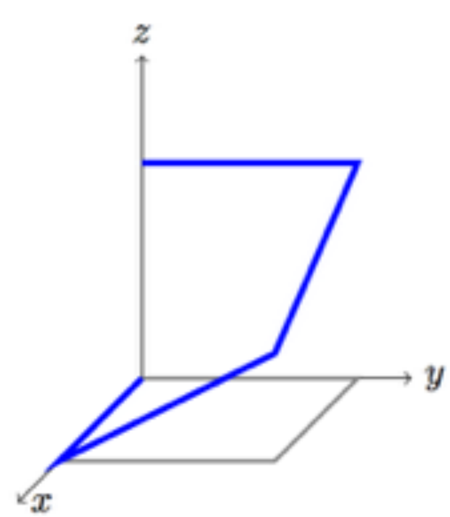

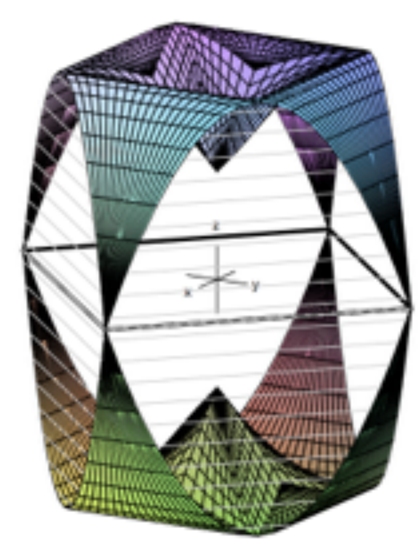

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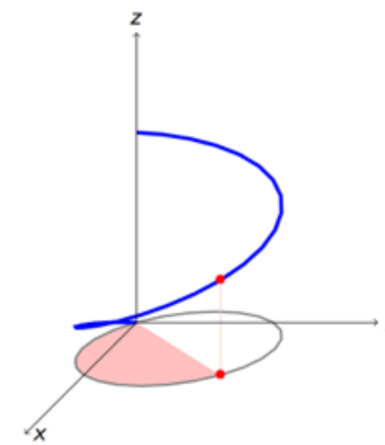

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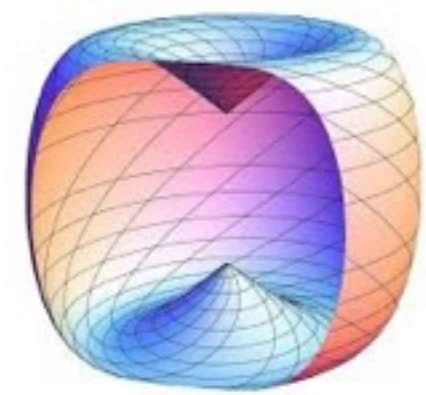

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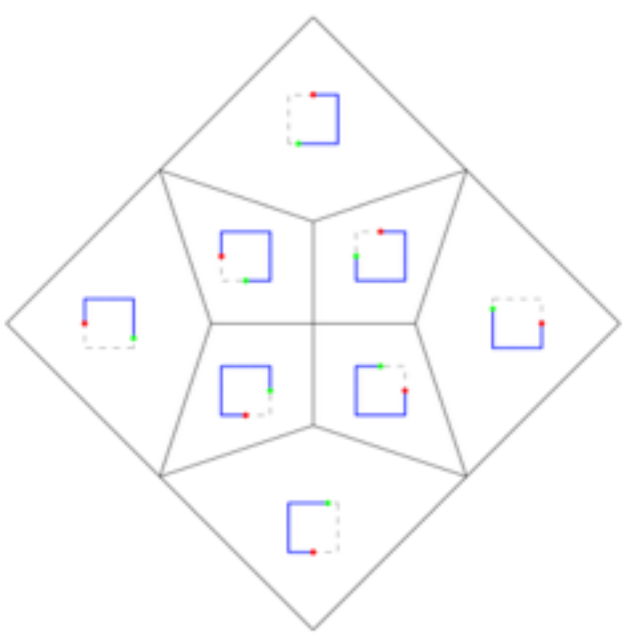

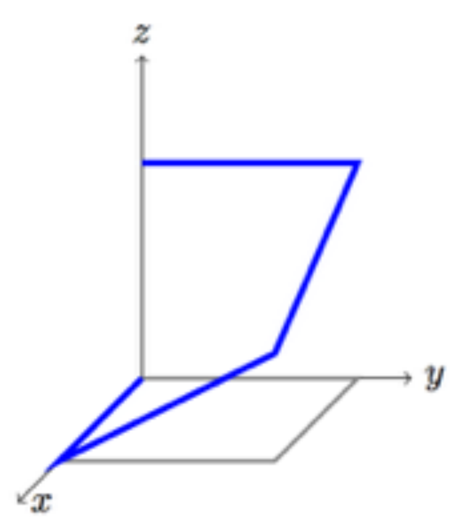

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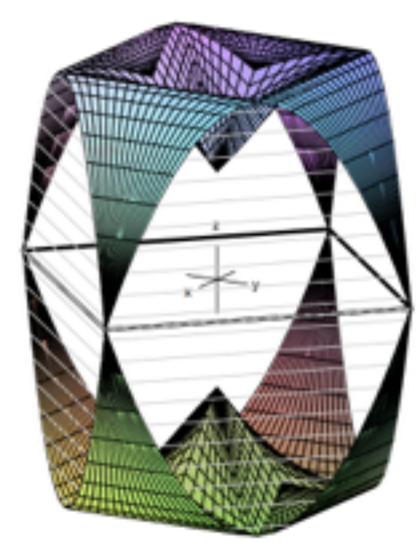

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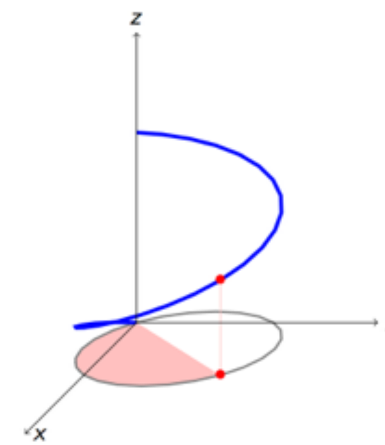

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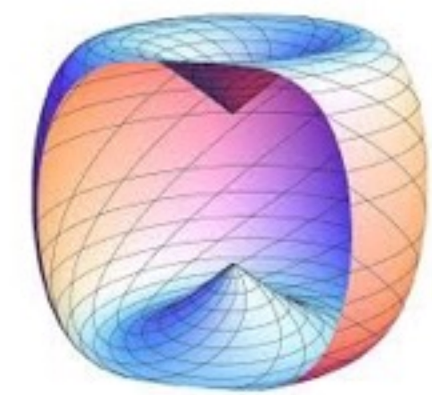

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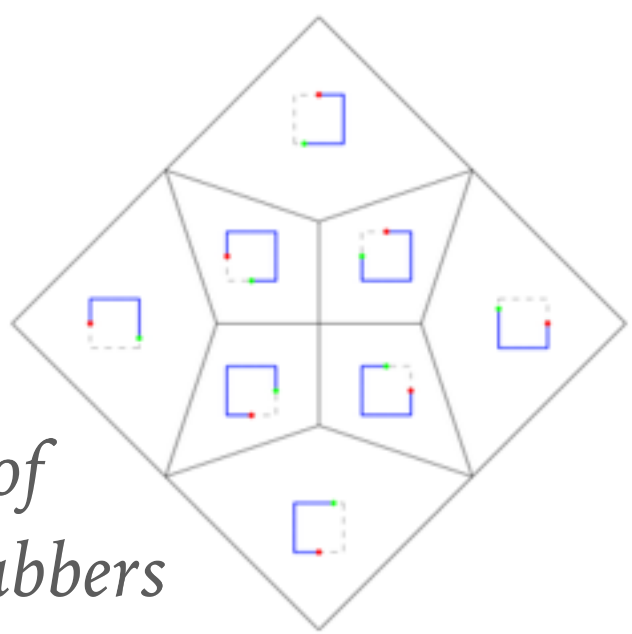

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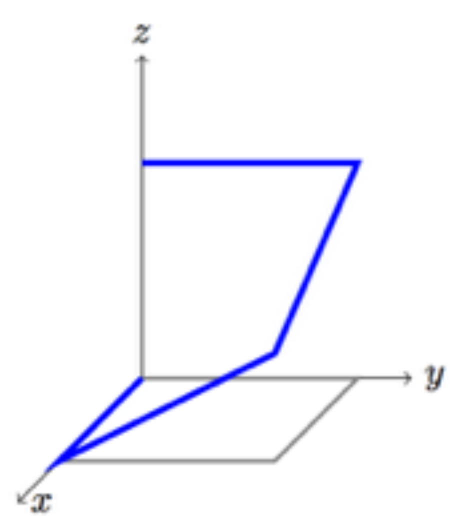

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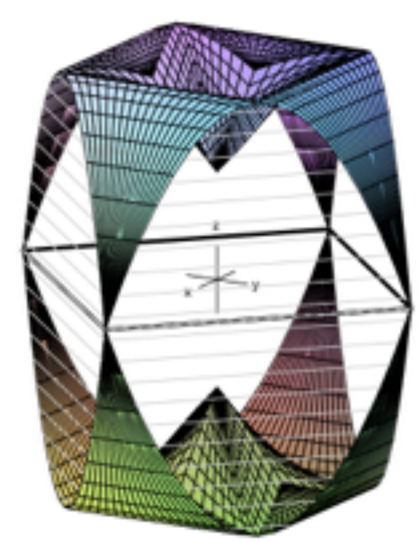

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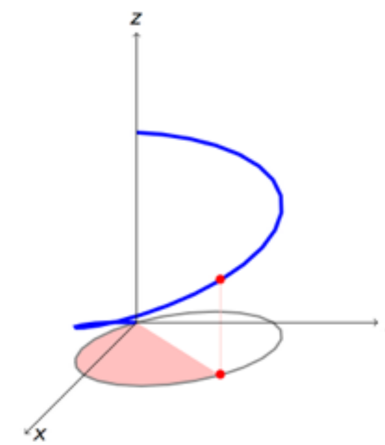

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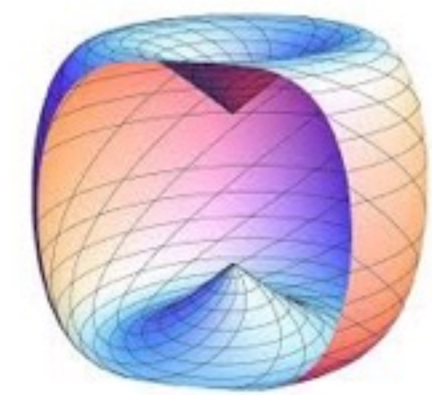

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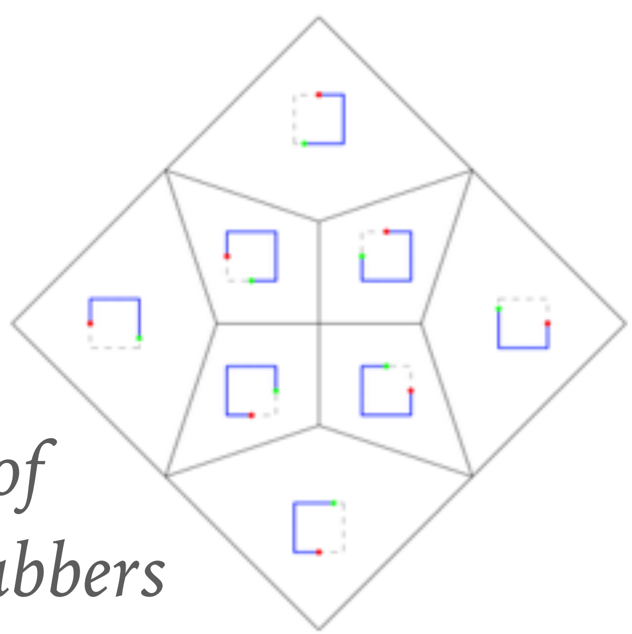

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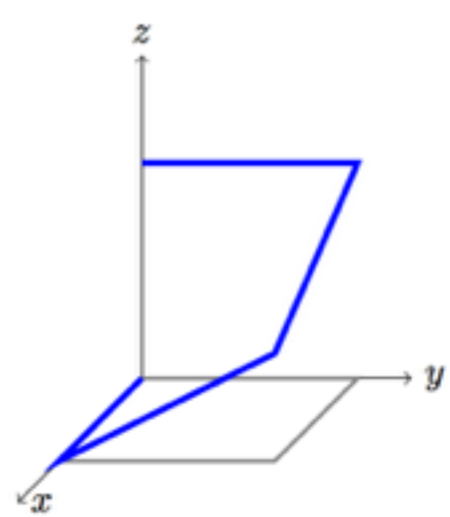

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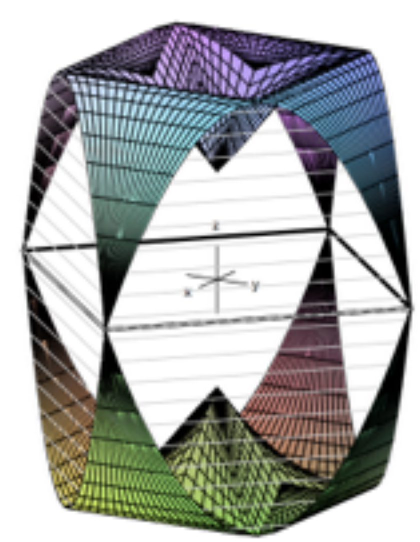

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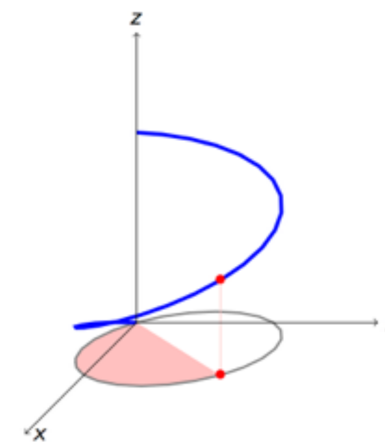

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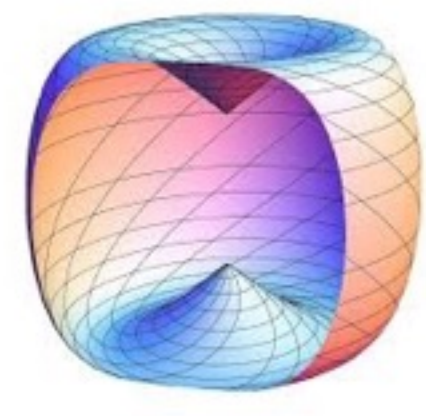

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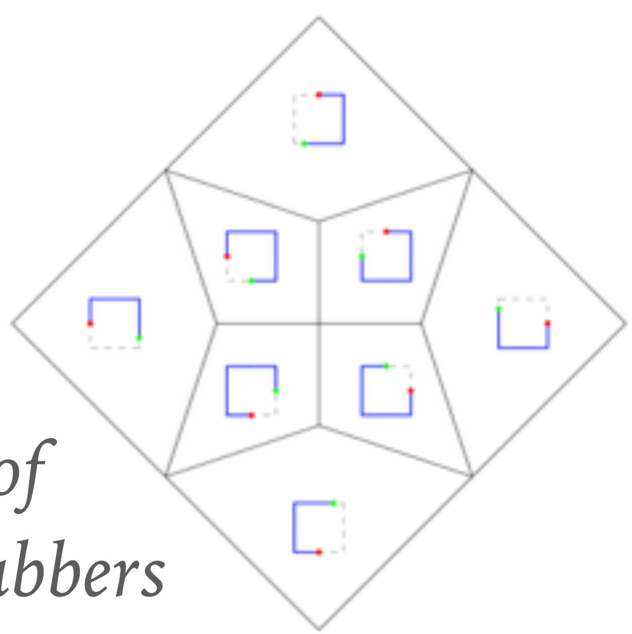

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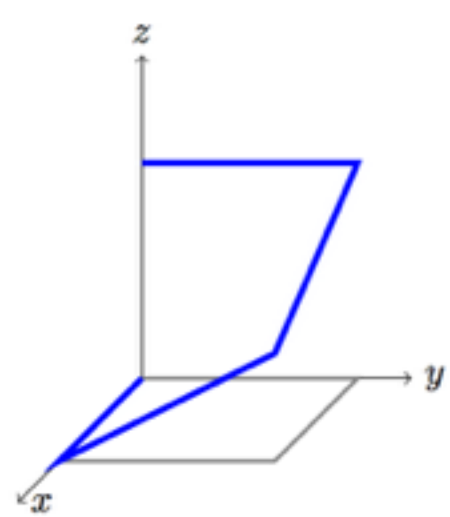

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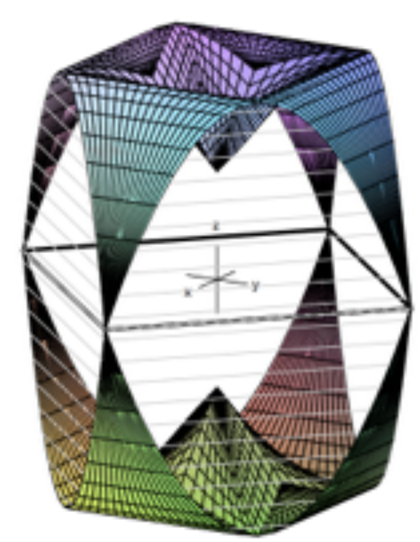

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note: walls are cut away to see inside —it's a topological sphere!

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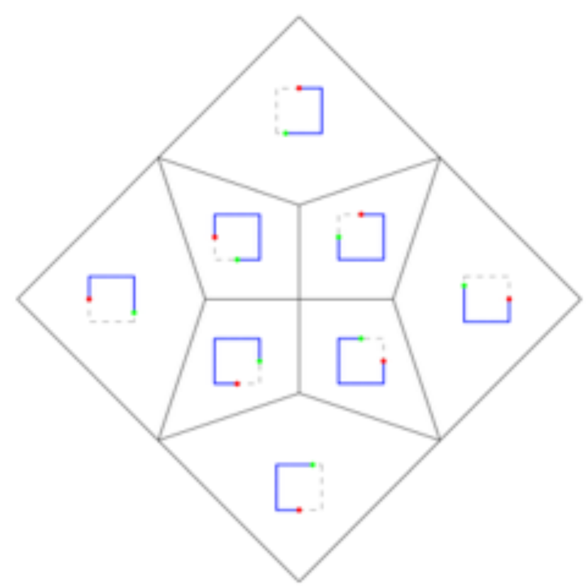

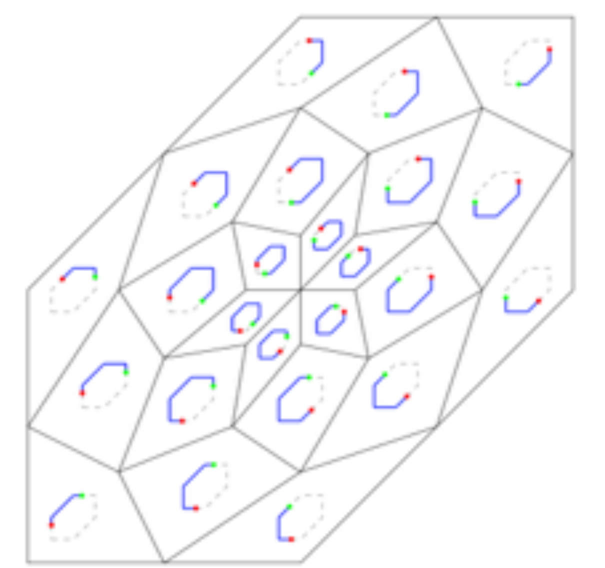

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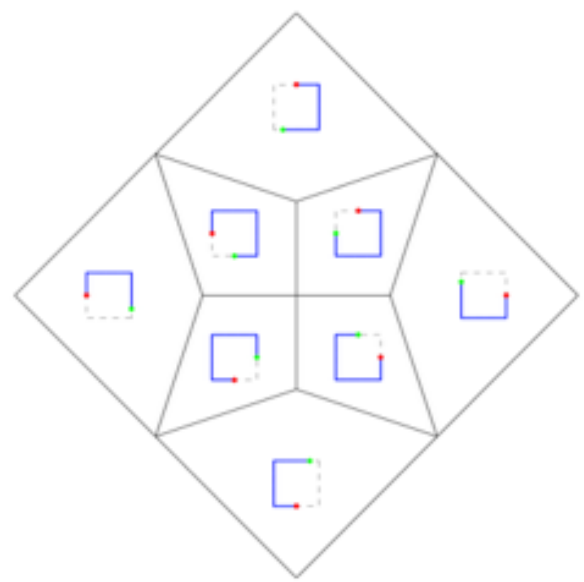

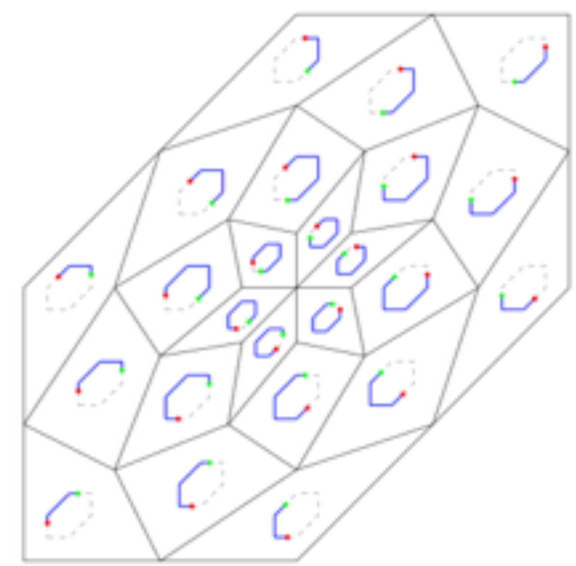

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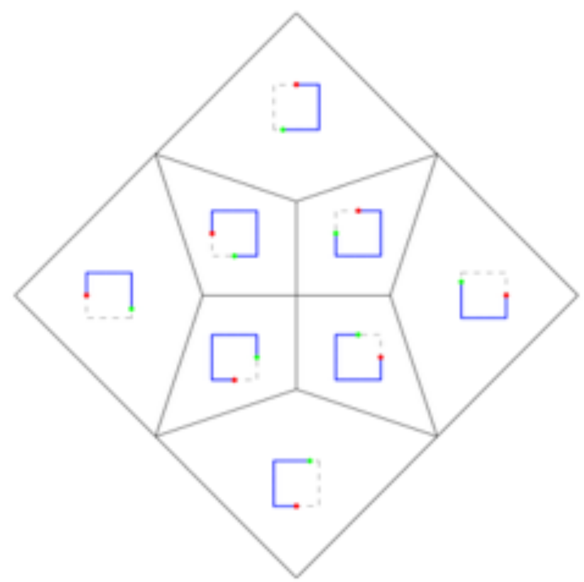

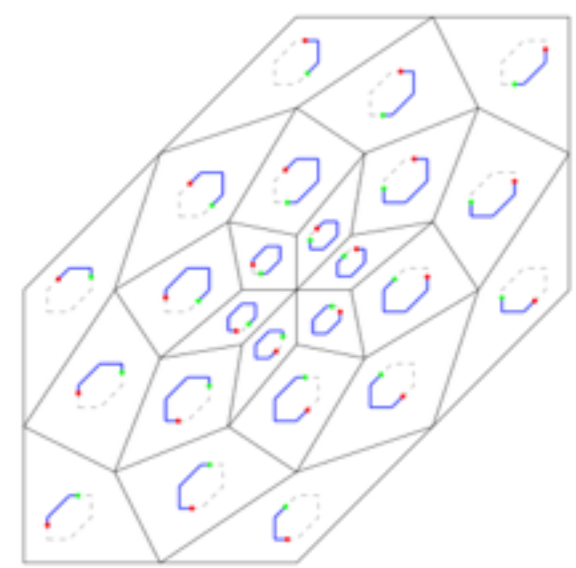

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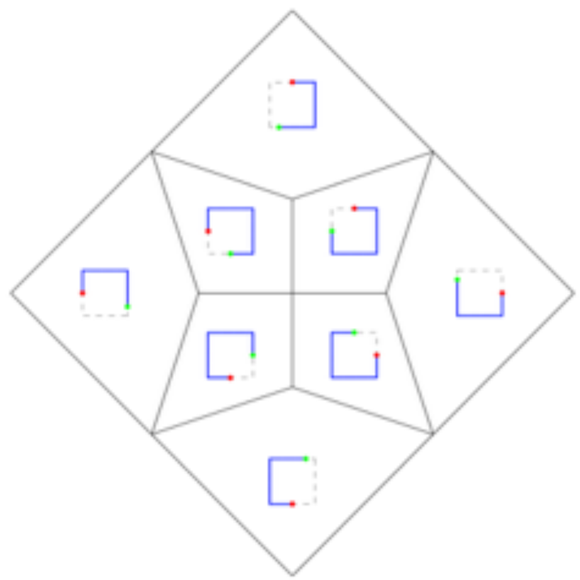

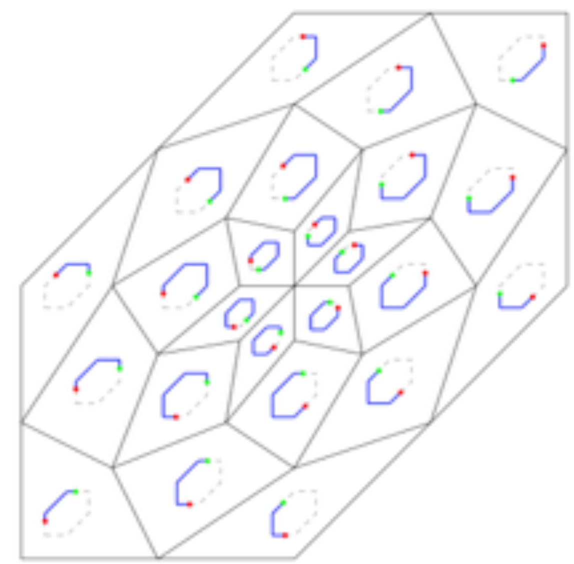

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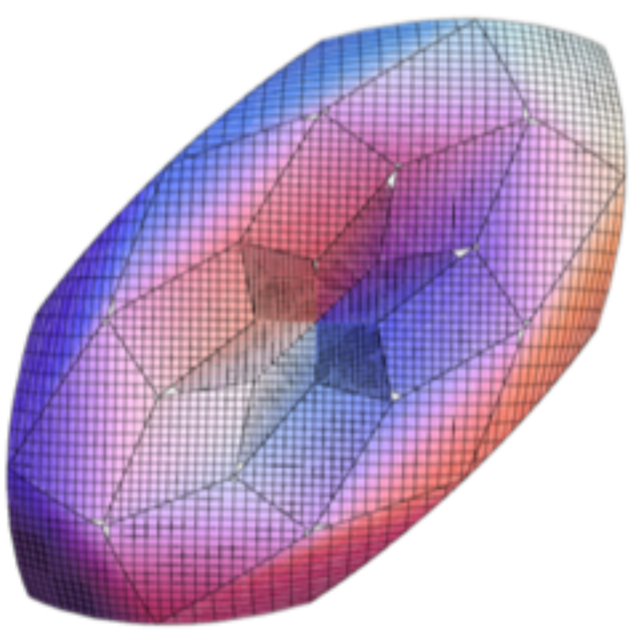

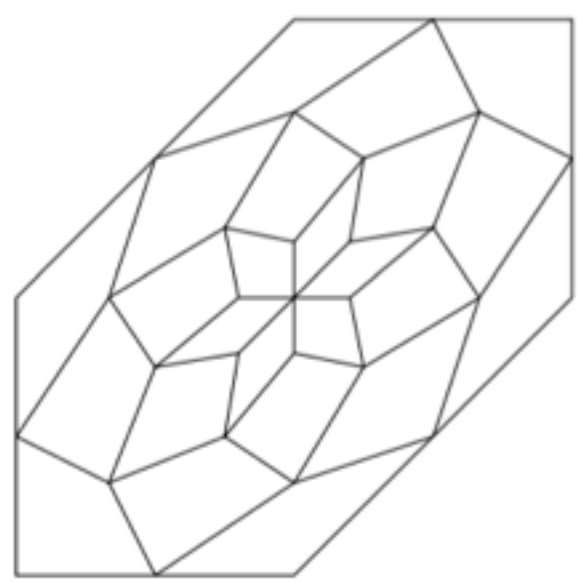

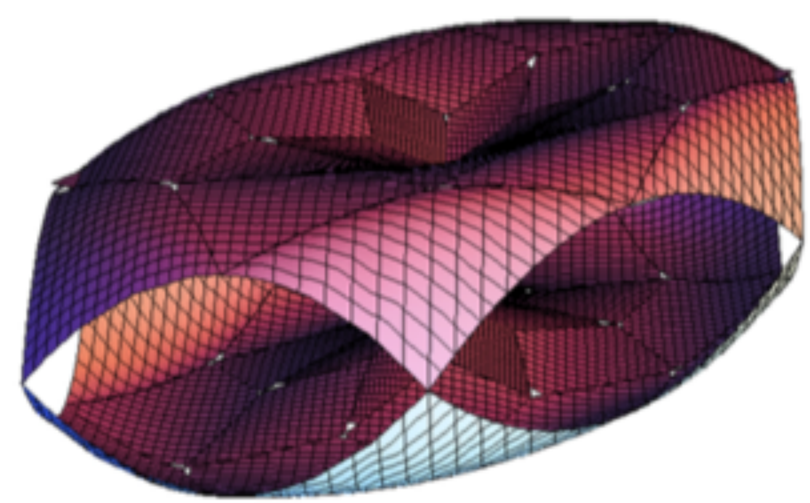

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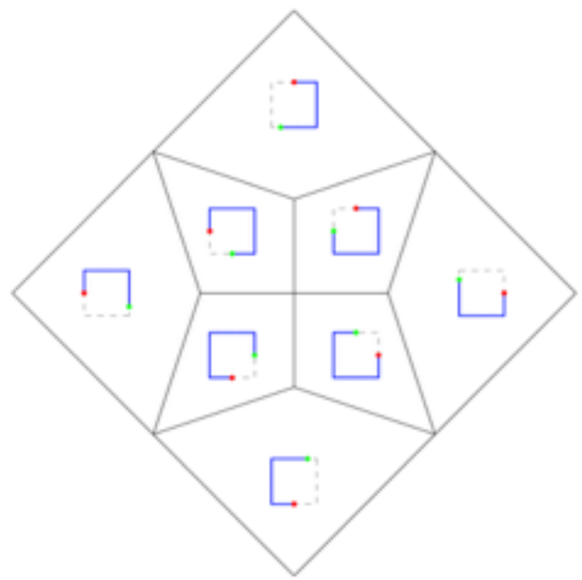

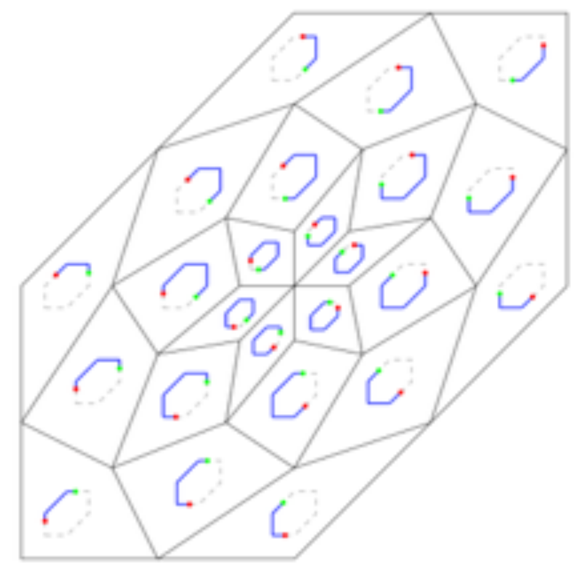

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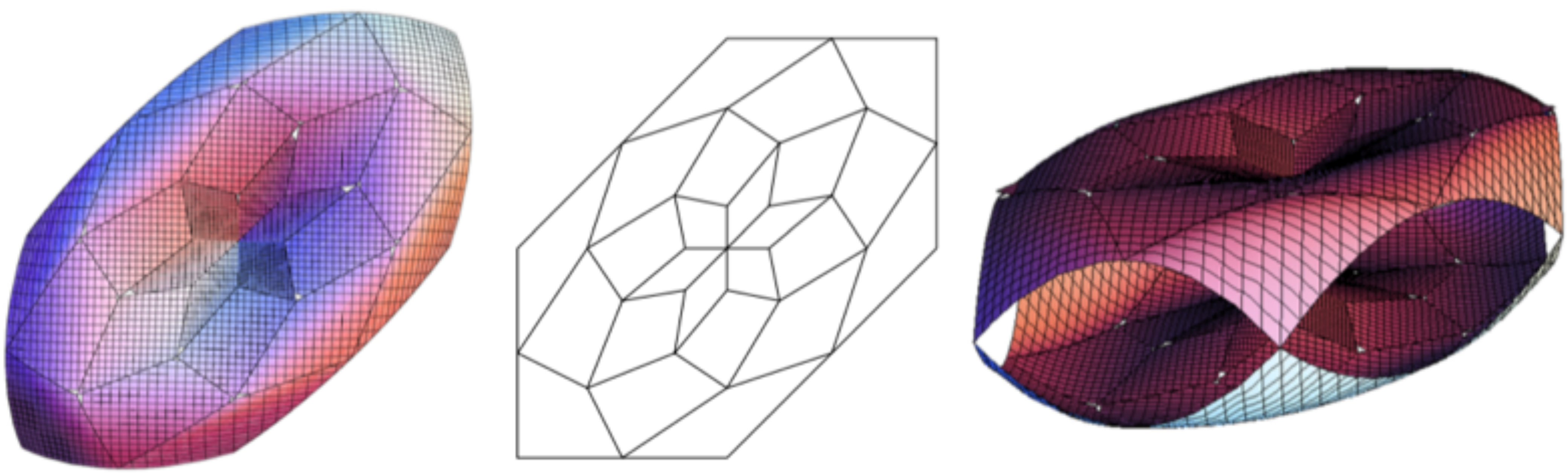

Plot of area enclosed by geodesics gives unit sphere as piecewise-quadratic graph

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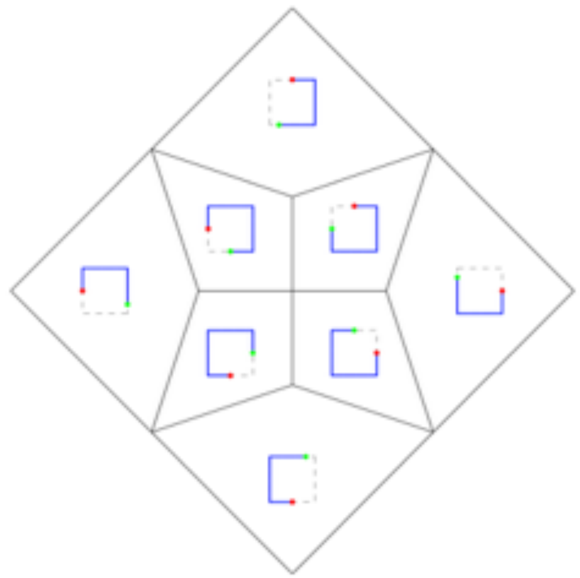

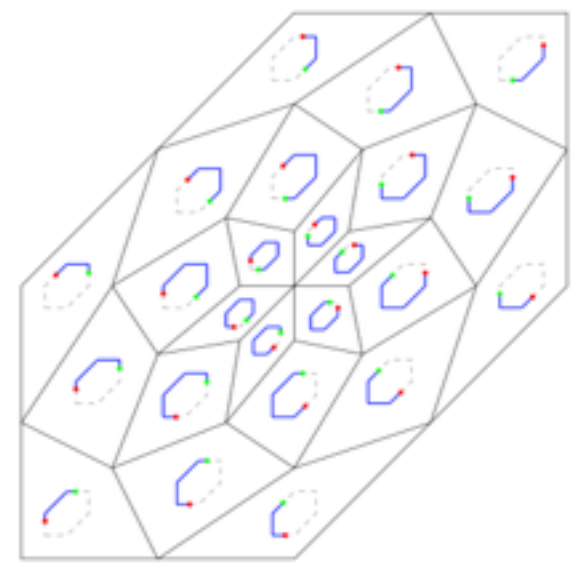

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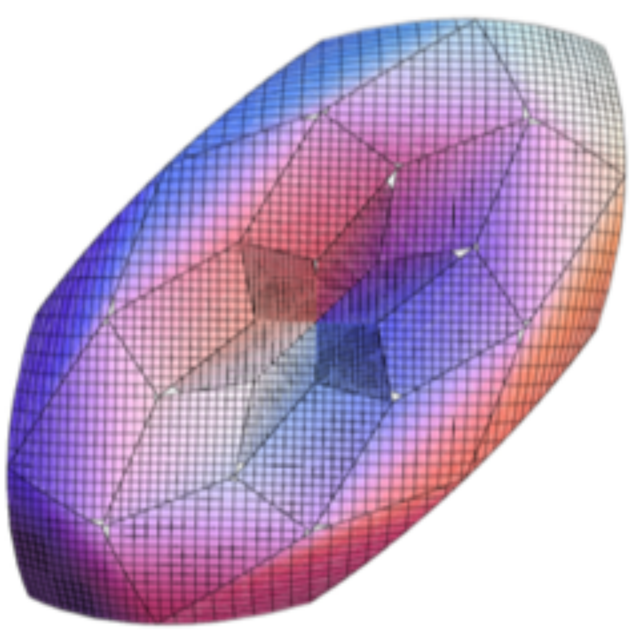

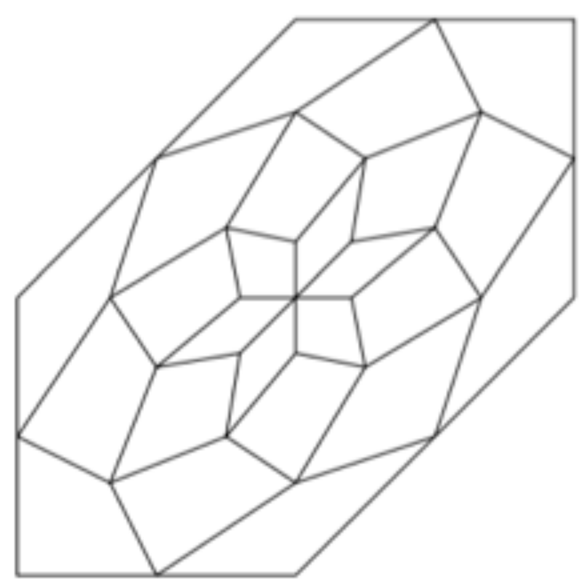

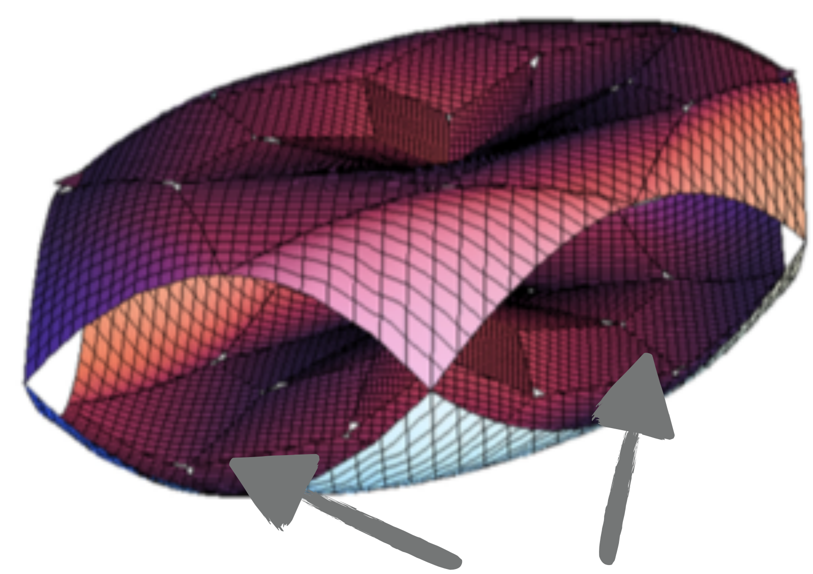

Plot of area enclosed by geodesics gives unit sphere as piecewise-quadratic graph

There are flat vertical "walls" coming from the beelines (range of areas for same endpoint)

SIDEBAR: WHERE DOES SUBRIEMANNIAN GEOMETRY COME UP?

Thurston's eight 3D "model geometries":

$$\mathbb{R}^3, \quad S^3, \quad \mathbb{H}^3, \quad S^2\times\mathbb{R}, \quad \mathbb{H}^2\times\mathbb{R}, \quad \text{Nil}, \quad \text{Sol}, \quad \text{and} \quad \widetilde{SL(2,\mathbb{R})}$$

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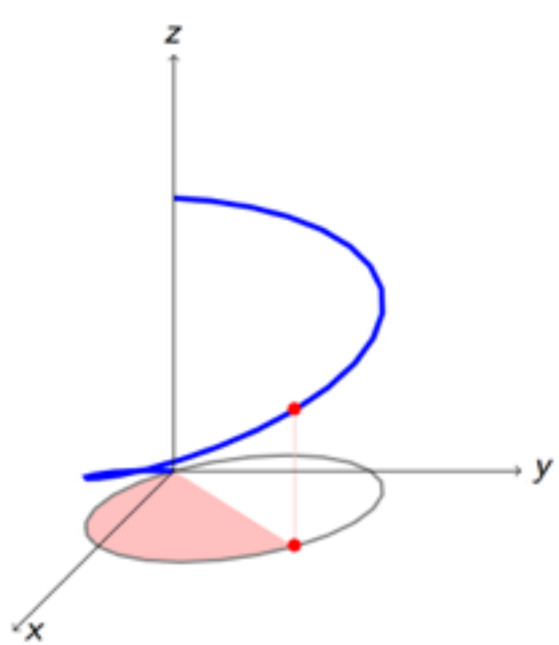

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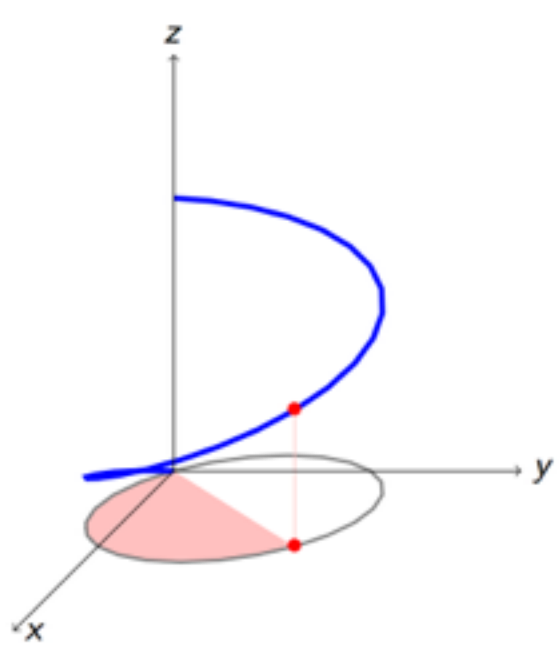

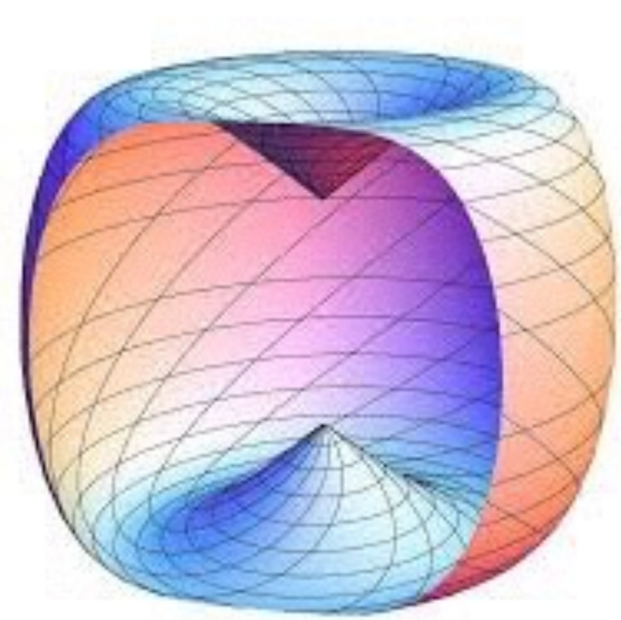

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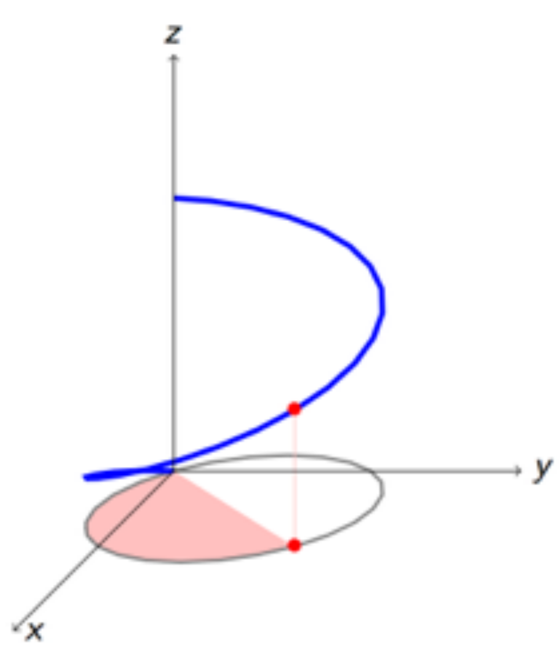

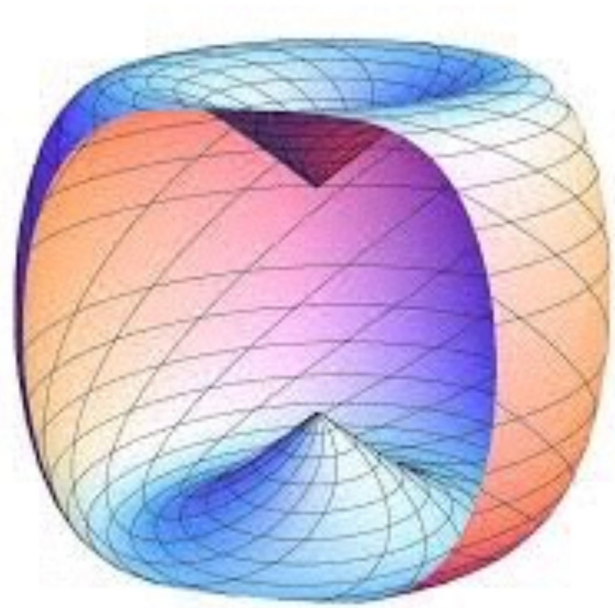

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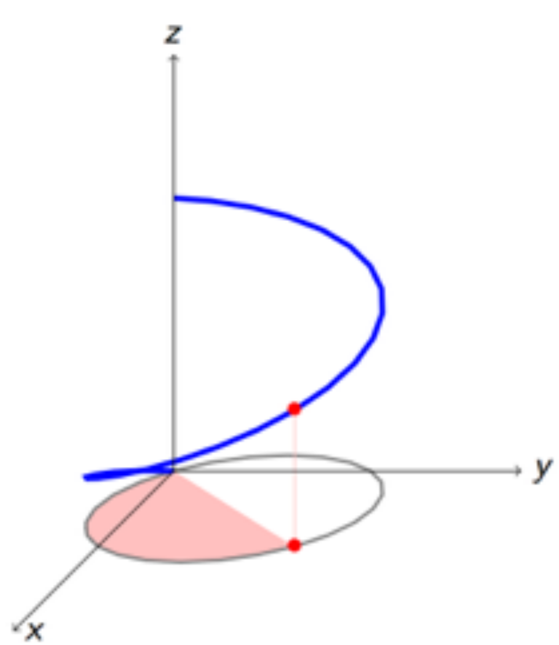
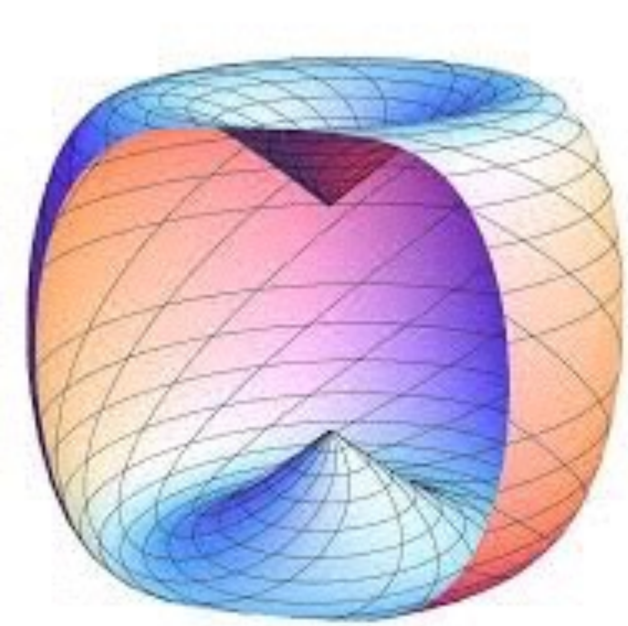
- Higher-dimensional $\mathbb{CH}^n$: horospheres are higher Heisenberg groups.





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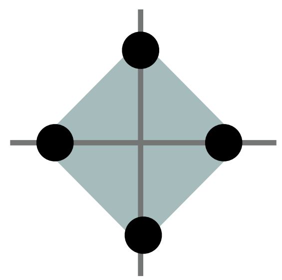

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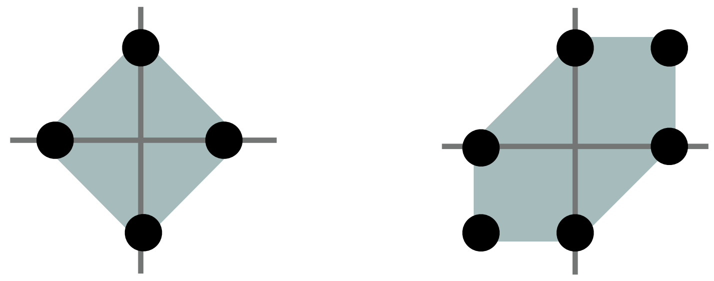

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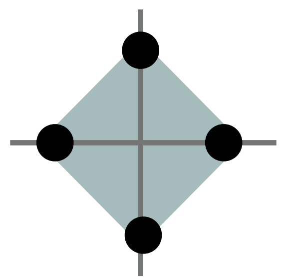

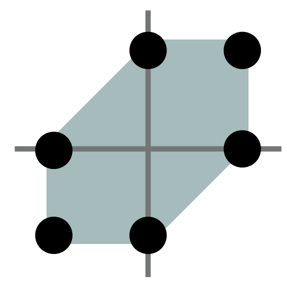

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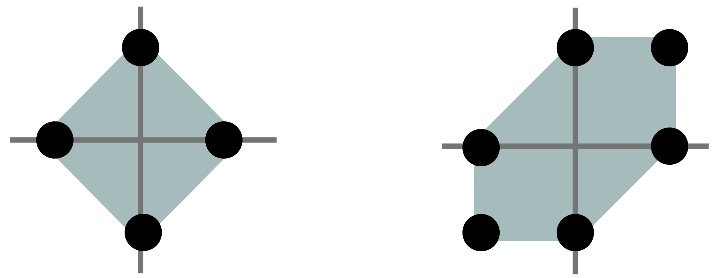

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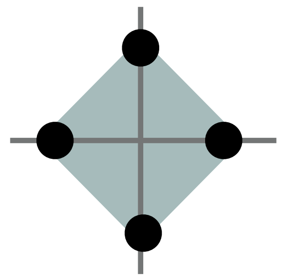

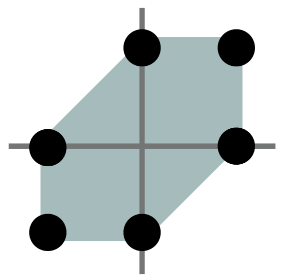

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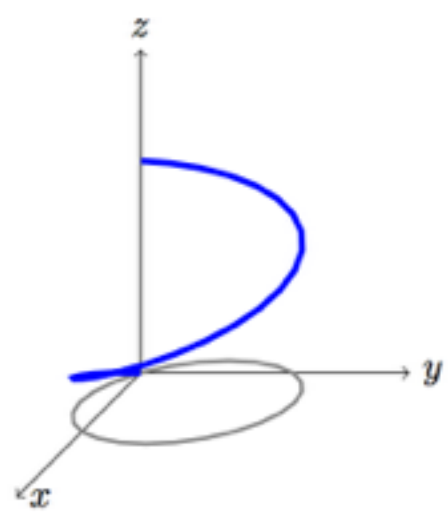

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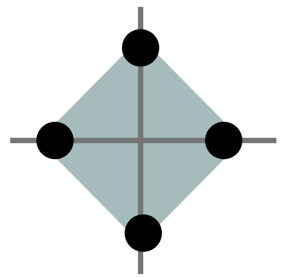

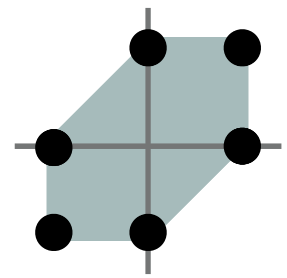

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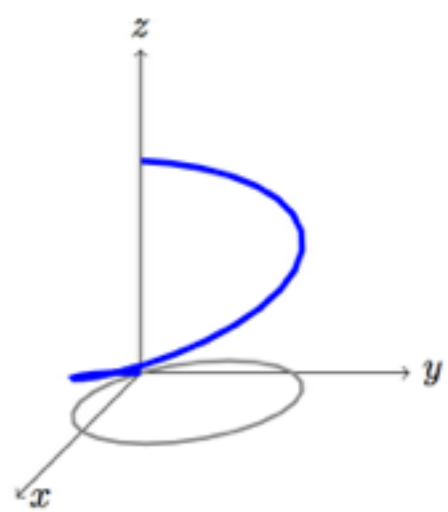

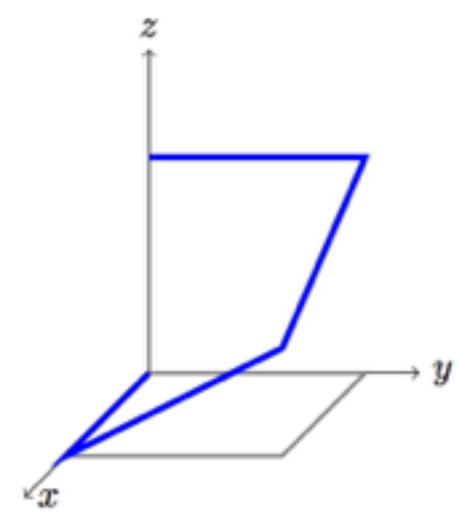

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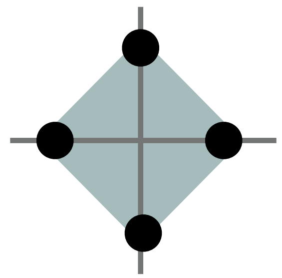

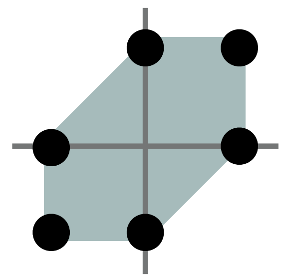

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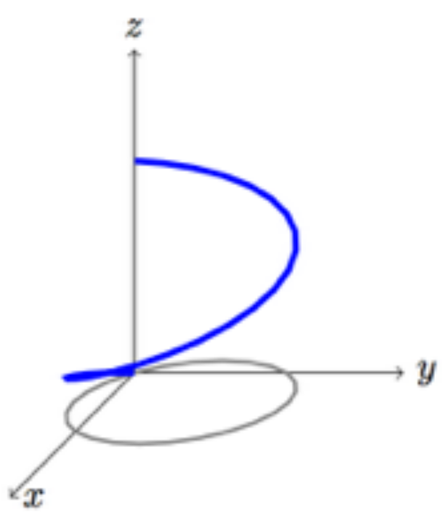

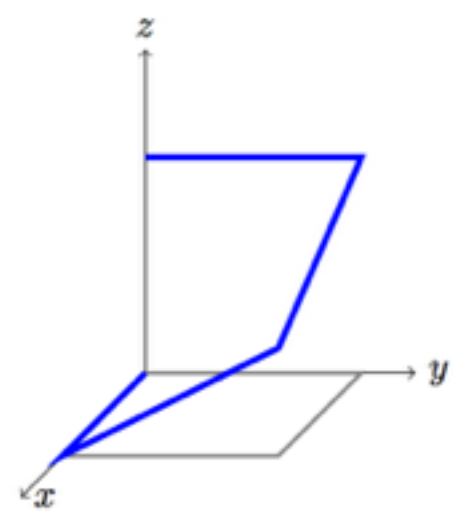

LATTICES IN THE LARGE: PANSU'S THEOREM

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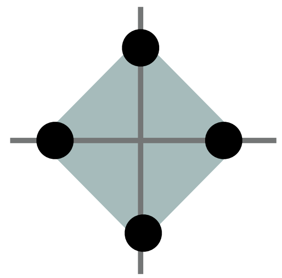

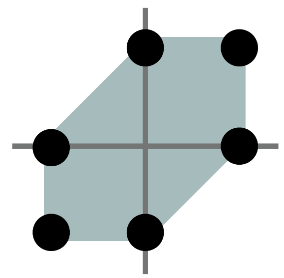

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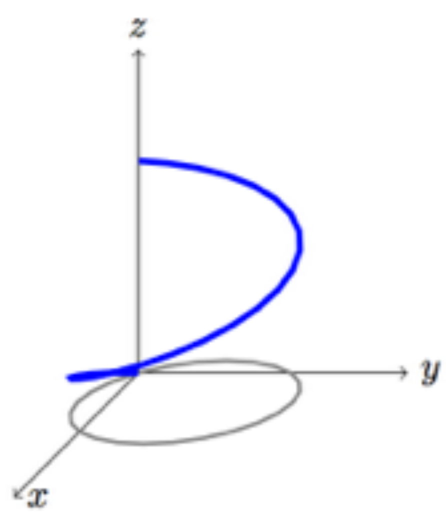

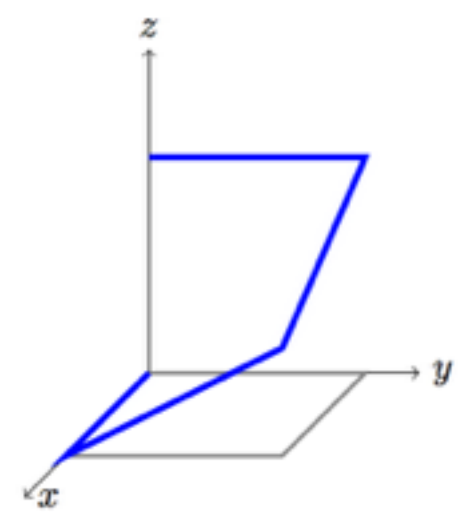

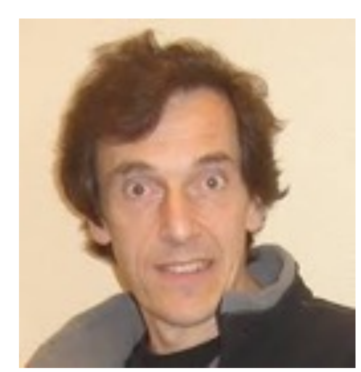

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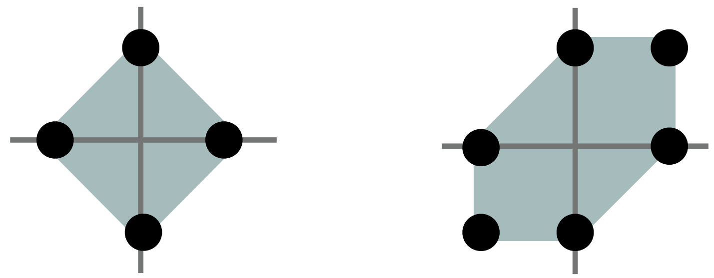

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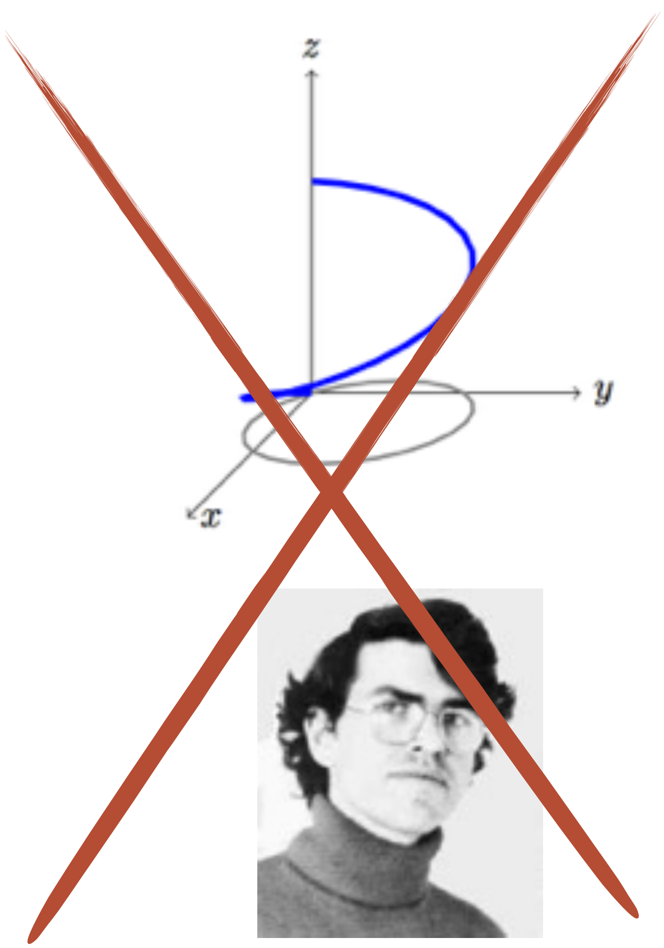

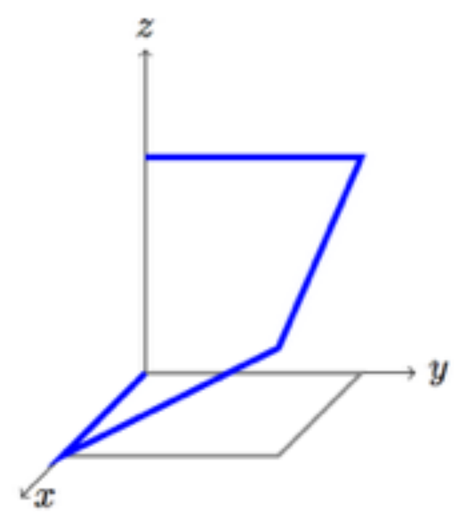

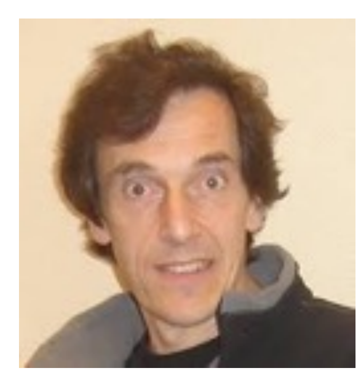

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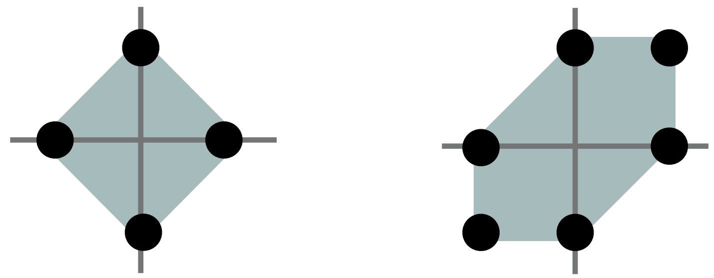

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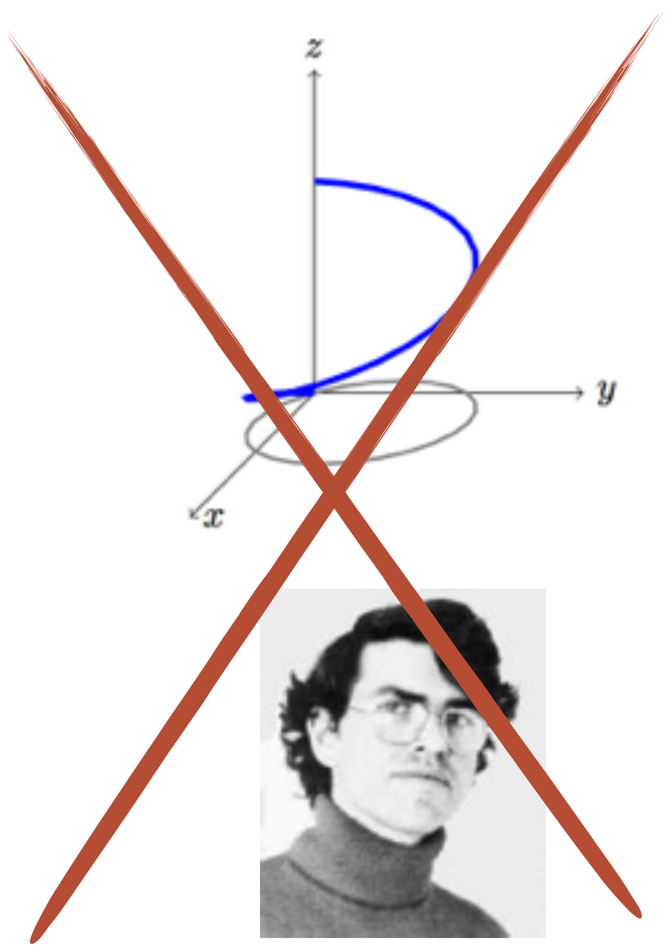

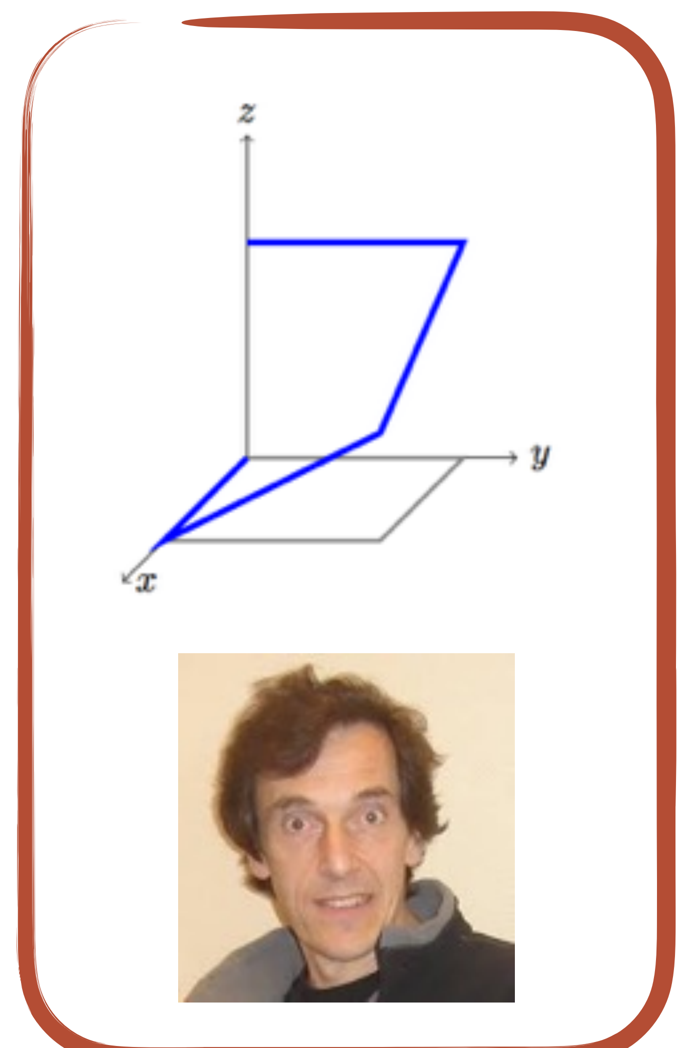

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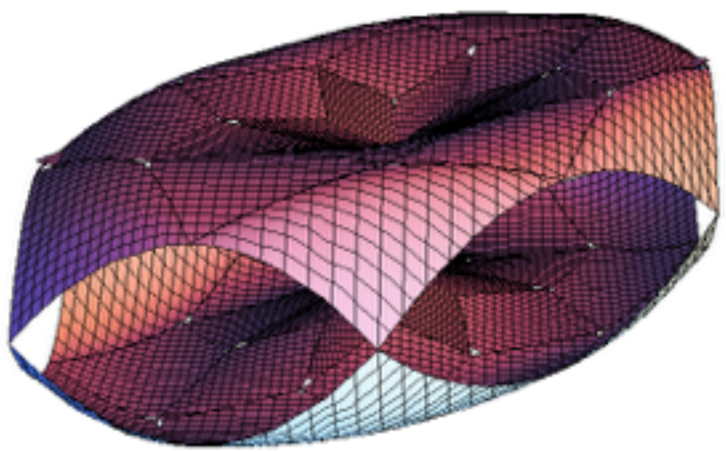

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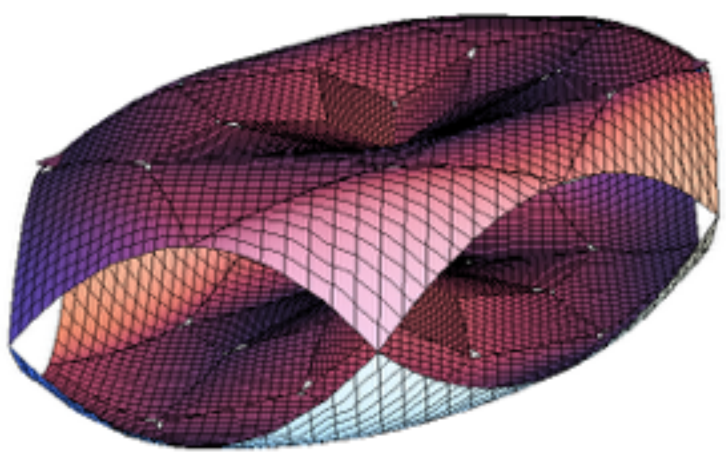

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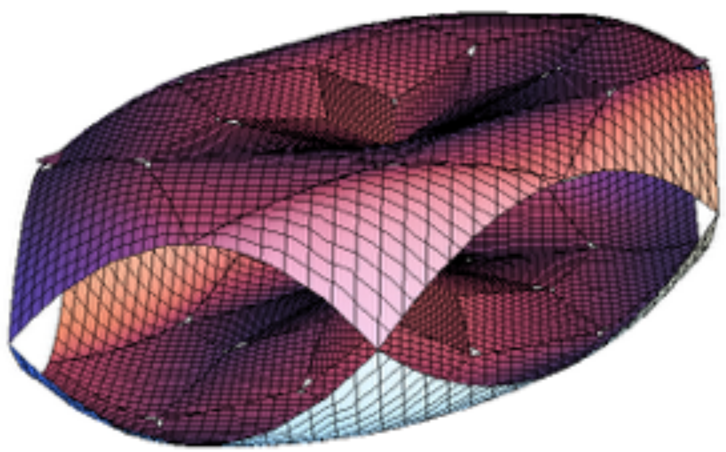

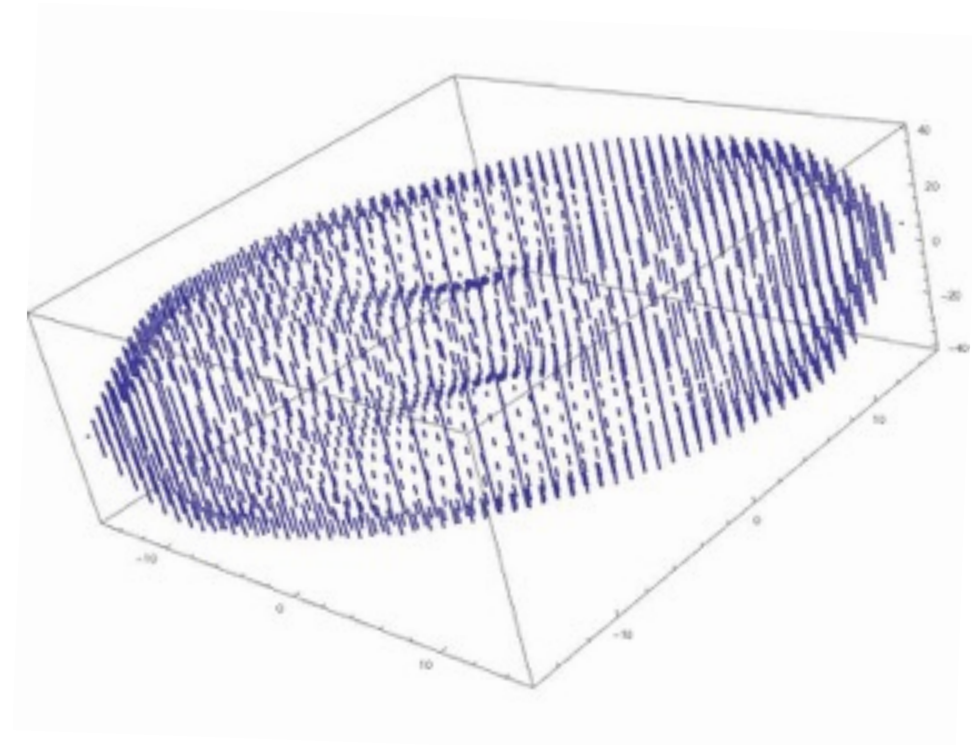

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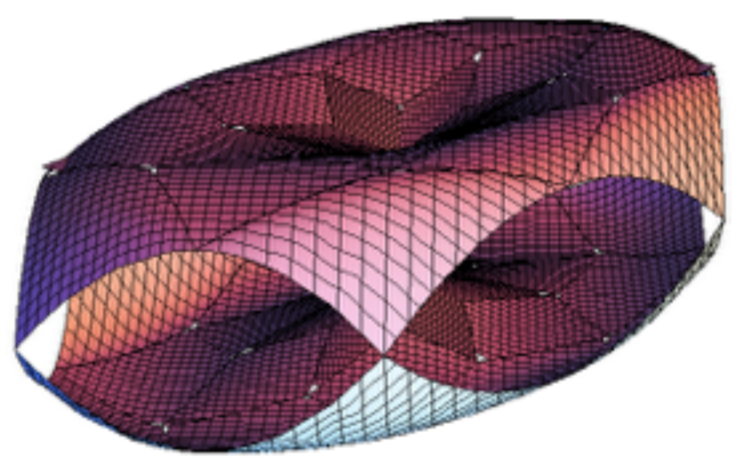

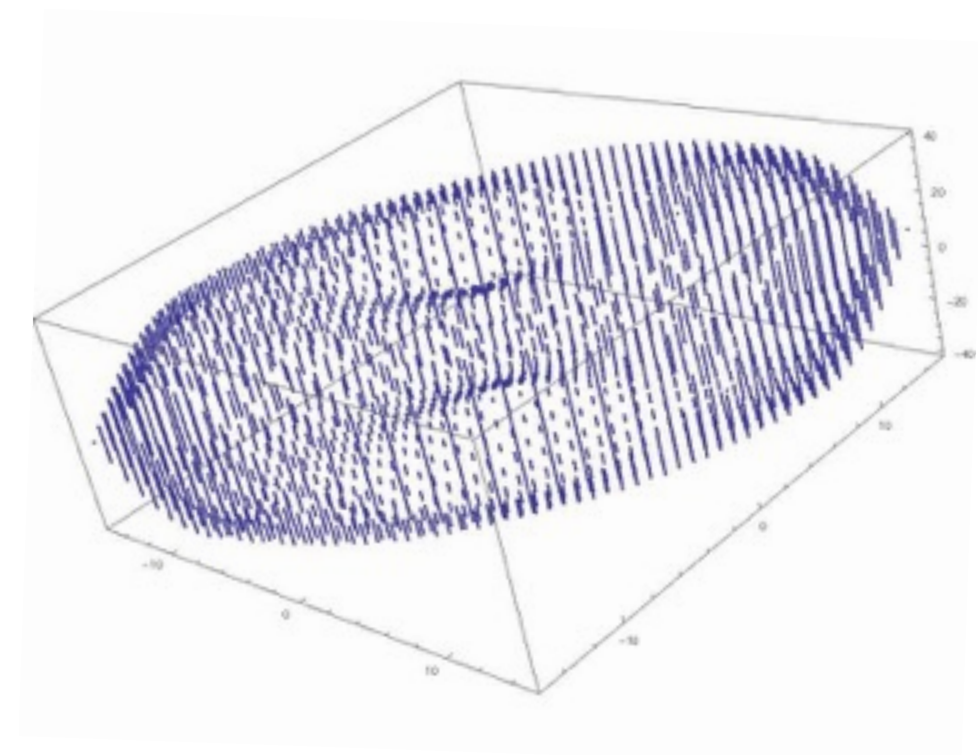

- Should still wonder: are CC *geodesics* good approximations of geodesics in the word metric?
- The CC group is divided into two parts(the beelines/walls and the area-grabbers/roof). What about the discrete group?

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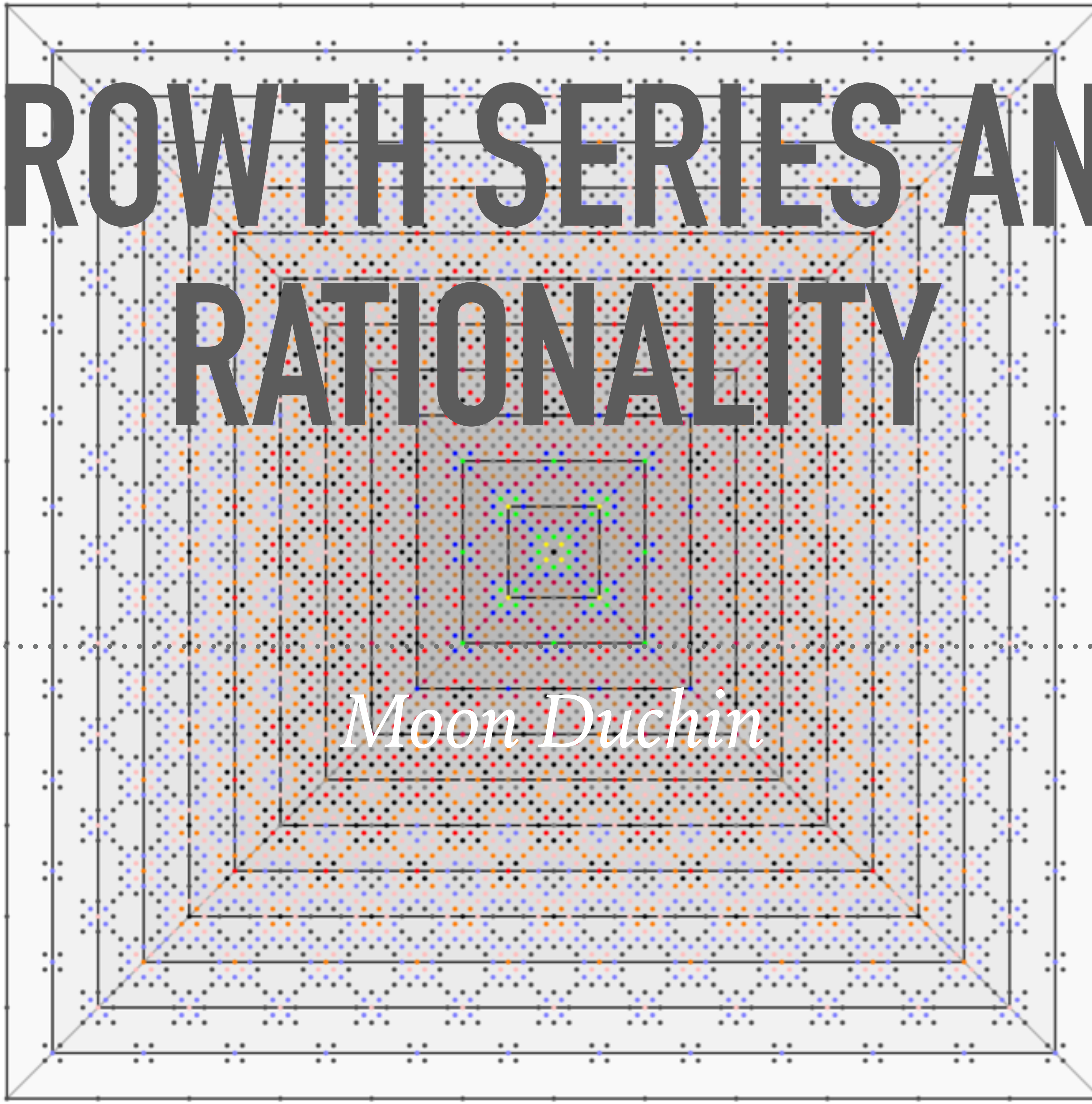
  - Visually, from a basepoint, space is divided up into two regimes (walls and roof), with rational proportion: many "p/q laws."

GROWTH SERIES AND RATIONALITY



Moon Duchin

GROWTH OF GROUPS

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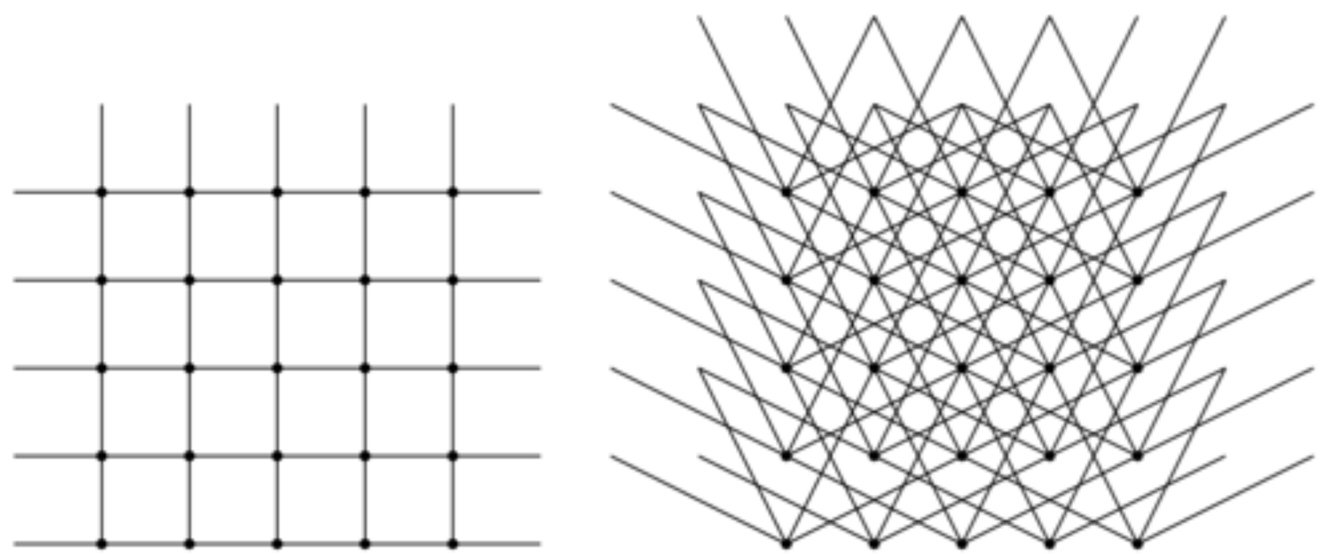

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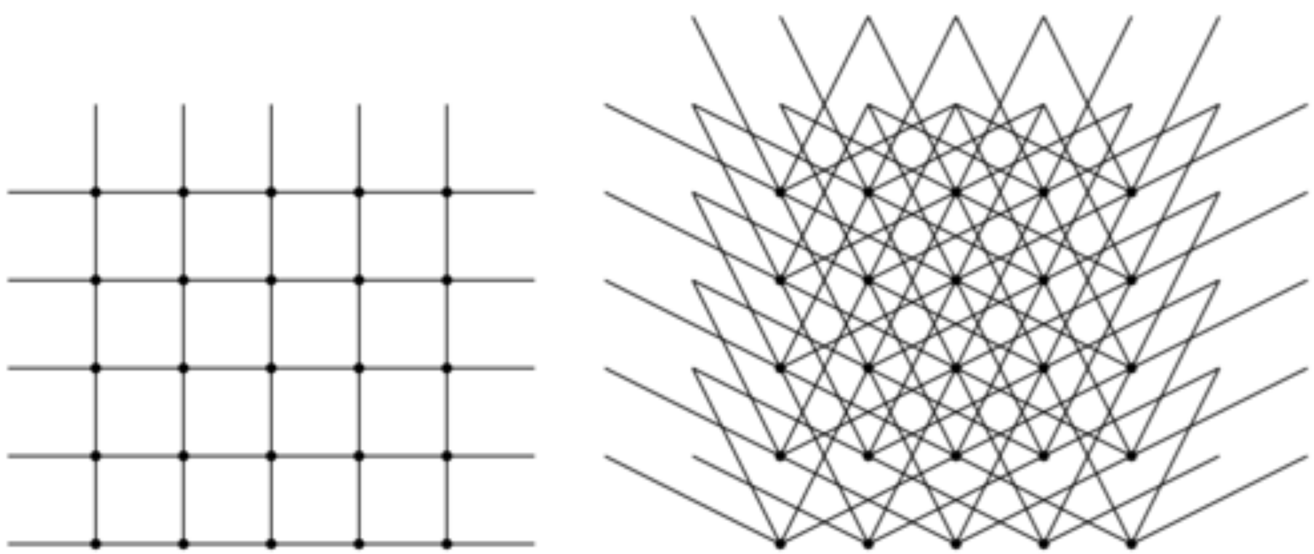

We can write $\beta_n = \#B_n$, $\sigma_n = \#S_n$ for the point count of balls and spheres in the word metric. As functions of n, these are called **growth functions** of (G,S) for a group G and generating set S.

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- That is: **nilpotent groups have polynomial growth**.

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$$\exists\, A \quad \text{s.t.} \quad f(t) \leq A g(At + A) + A \quad \forall t.$$

If $f \sim g$ means $f \preccurlyeq g$, $g \preccurlyeq f$, then $[\sigma]$ is a quasi-isometry invariant.

Theorem (Bass-Guivarc'h 1970s): *if LCS of a nilpotent group has quotients* $G_i/G_{i+1} = \mathbb{Z}^{d_i} \oplus \text{finite}$ *, then* $\beta(n) \sim n^d$ *for* $d = \sum i\, d_i$.

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  Theorem (Stoll 1996): H_5 *has rational growth in one generating set but transcendental in another!*

SUMMARY OF RATIONALITY RESULTS

For all S	For at least one S	For no S
hyperbolic groups	some automatic groups	unsolvable word problem
virtually abelian groups	Coxeter groups, standard S	intermediate growth
Heisenberg group H ←	~~H, standard S~~	
	H_5, cubical S	
	$BS(1, n)$, standard S	

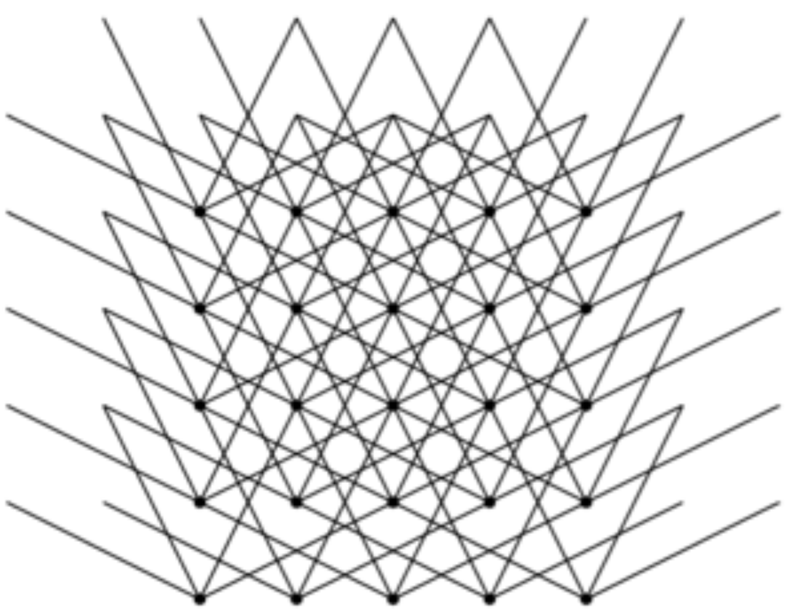
- Many more in middle category: some more *BS(p,q)* examples plus "higher BS groups," quotients of triangular buildings, some amalgams and wreath products, some solvable groups, relatively hyperbolic groups, …

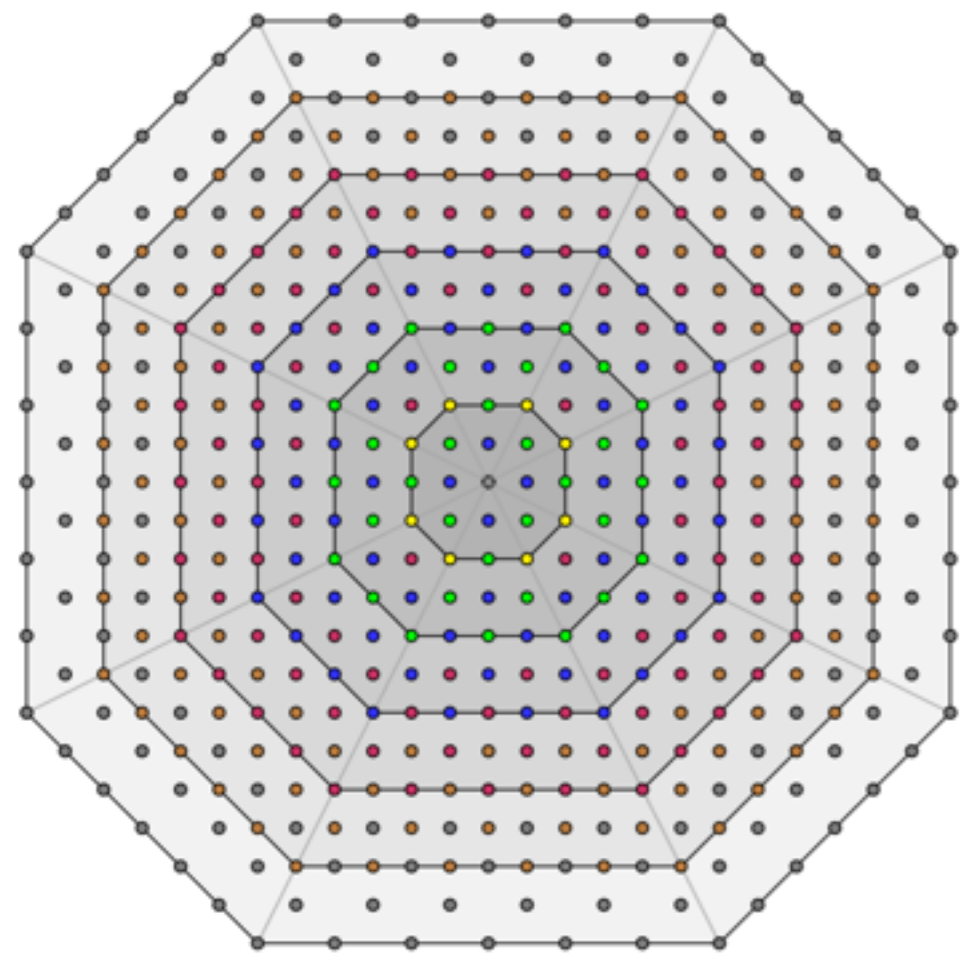
CANNON 1984, BEING ZEN ABOUT GROUPS

- "We shall… show that the **global combinatorial structure** of such groups is particularly simple in the sense that their Cayley group graphs (Dehn Gruppenbilder) have descriptions by linear recursion. We view this latter result as a promising generalization of small cancellation theory… The result also indicates that cocompact, discrete hyperbolic groups **can be understood globally** in the same sense that the integers $\mathbb{Z}$ can be understood: feeling, as we do, that we understand the simple linear recursion $n \to n+1$ in $\mathbb{Z}$, we **extend our local picture** of $\mathbb{Z}$ **recursively in our mind's eye toward infinity**. One obtains a global picture of the arbitrary cocompact, discrete hyperbolic group G in the same way: first, one discovers the local picture of G, then the recursive structure of G by means of which copies of the local structure are integrated."

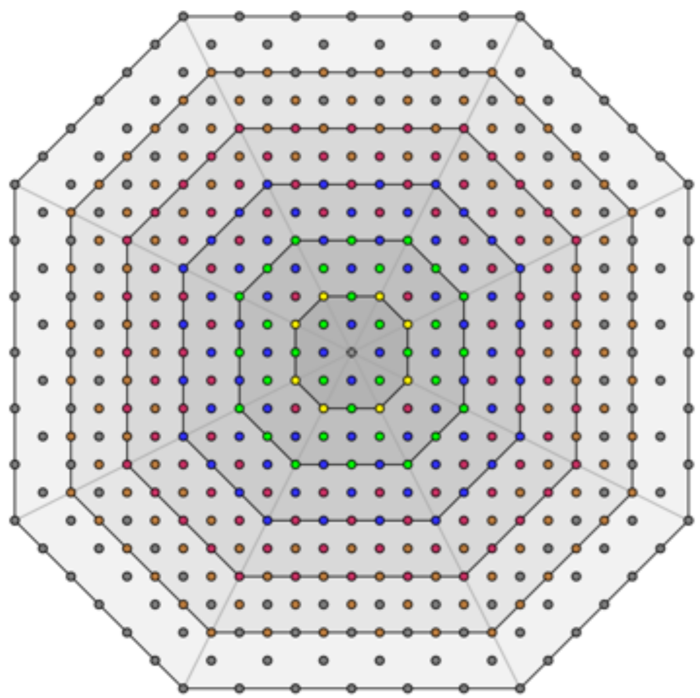
RELATING GROUP GROWTH TO LATTICE COUNTING

Let's go back and see why lattice counts and Ehrhart polynomials were related to growth functions for $\mathbb{Z}^2$. You can get a coarse estimate of β_n by figuring out the shape of the cloud of points B_n and counting lattice points inside it.



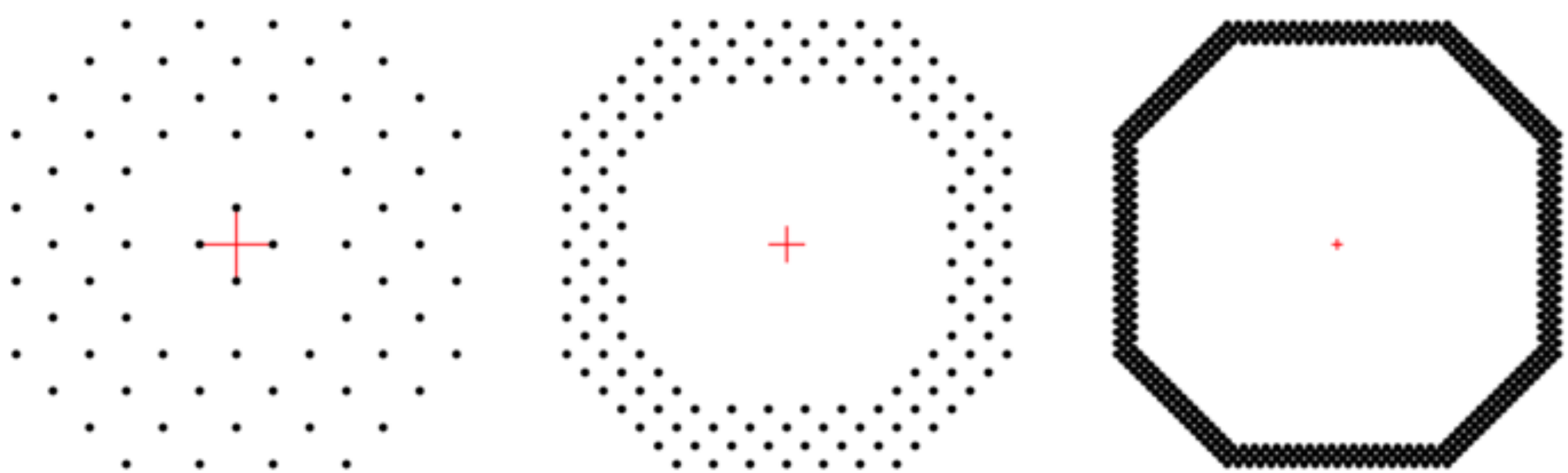


WORD METRICS HAVE "LIMIT SHAPES" YOU CAN COUNT WITH



To see that you get an accurate first-order estimate from the Ehrhart polynomial, it suffices to show that almost all lattice points in the nth dilate are reached in n steps.

This works well here; in general, the large spheres very closely resemble an annular shell at the boundary of your defining polygon.



S_3 S_6 S_{20}

GROUP GROWTH MEETS LATTICE COUNTING

(G,S)	$\beta_n \quad (n \gg 1)$	Ω	$G_\Omega(n)$
$(\mathbb{Z}, \text{std})$	$2n+1$	—	$2n+1$
$(\mathbb{Z}^2, \text{std})$	$2n^2+2n+1$		$2n^2+2n+1$
$(\mathbb{Z}^2, \text{hex})$	$3n^2+3n+1$		$3n^2+3n+1$
$(\mathbb{Z}^2, \text{chess})$	$14n^2-6n+5$		$14n^2+6n+1$
$(\mathbb{Z}^3, \text{std})$	$\frac{(2n+1)(2n^2+2n+3)}{3}$		$\frac{(2n+1)(2n^2+2n+3)}{3}$
(F_2, std)	$2\cdot 3^n - 1$	?	?

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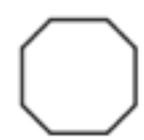
$(\mathbb{Z}^2, \text{chess})$	$14n^2-6n+5$		$14n^2+6n+1$
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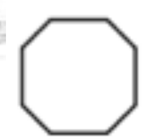
$(\mathbb{Z}^2, \text{chess})$	$14n^2-6n+5$		$14n^2+6n+1$
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COUNTING IN THE HEISENBERG GROUP

Theorem (Shapiro, Benson 1980s): *The spherical growth of $H(\mathbb{Z})$, std is*

$$\sigma_n = (31n^3 - 57n^2 + 105n + c_n)/18,$$

where c_n=-7, -14, 9, -16, -23, 18, -7, -32, 9, 2, -23, 0, repeating with period 12, for $n \geq 1$.

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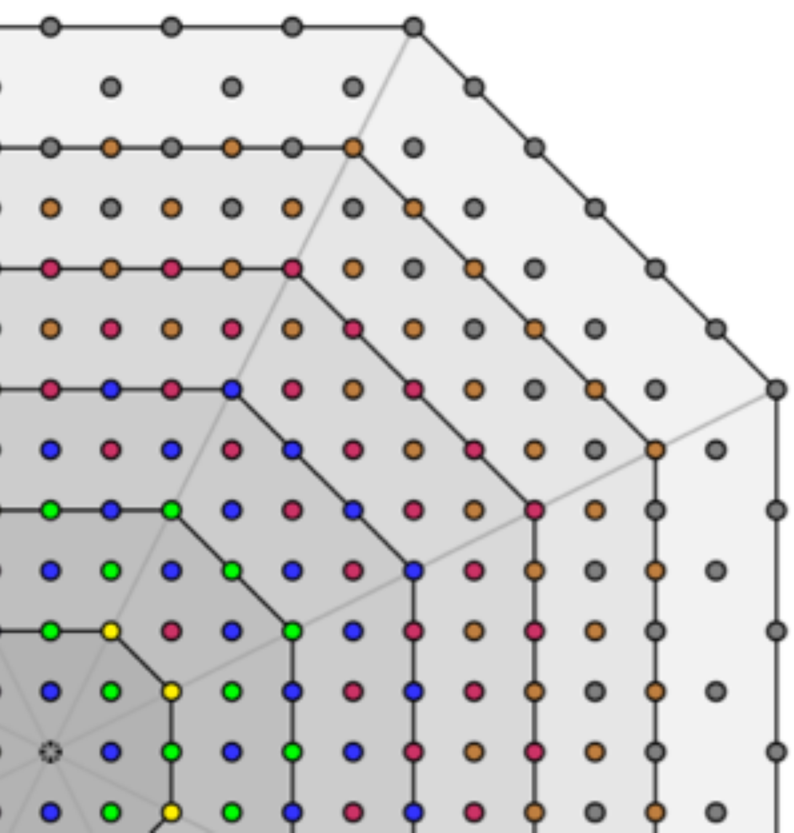
(not just bounded above and below like $An^3 \leq \sigma_n \leq Bn^3$, which is classical)

GAME PLAN FOR HEISENBERG PANRATIONALITY

- We produce a finite collection of languages that we call *shapes* and *patterns* that surject onto H.
- We show that there are "rational competitions" that determine a single shape or pattern as the "winner" for each group element.
- We show that enumerating the winning spellings by length is a rational function for each shape and pattern.
- Conclusion: overall growth function is a sum of finitely many rational functions, so it is rational.

MOTIVATING EXAMPLE: KNIGHTS ON AN INFINITE CHESSBOARD



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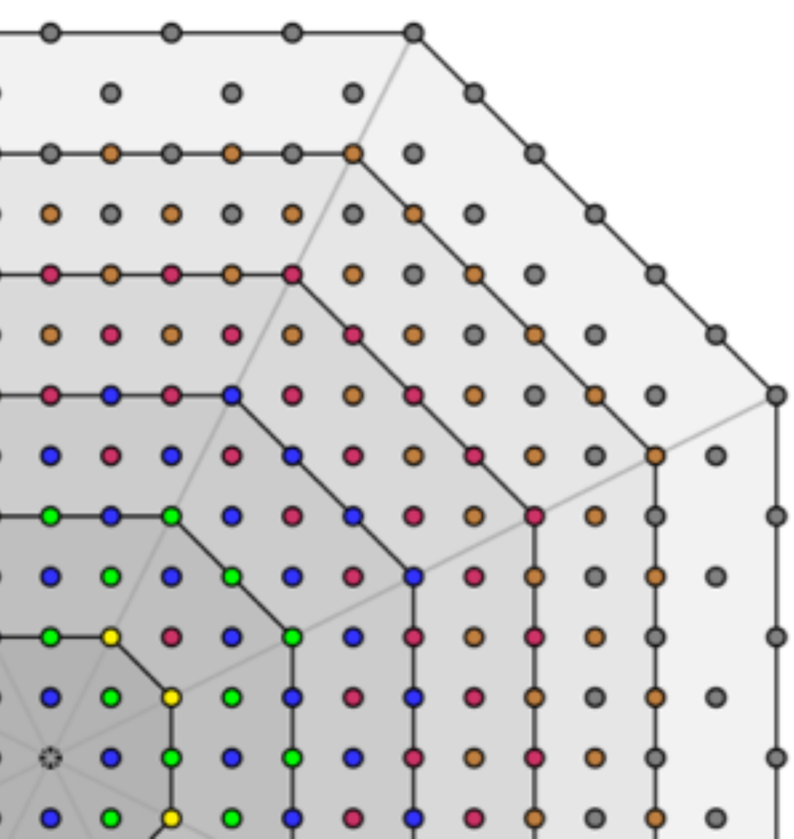

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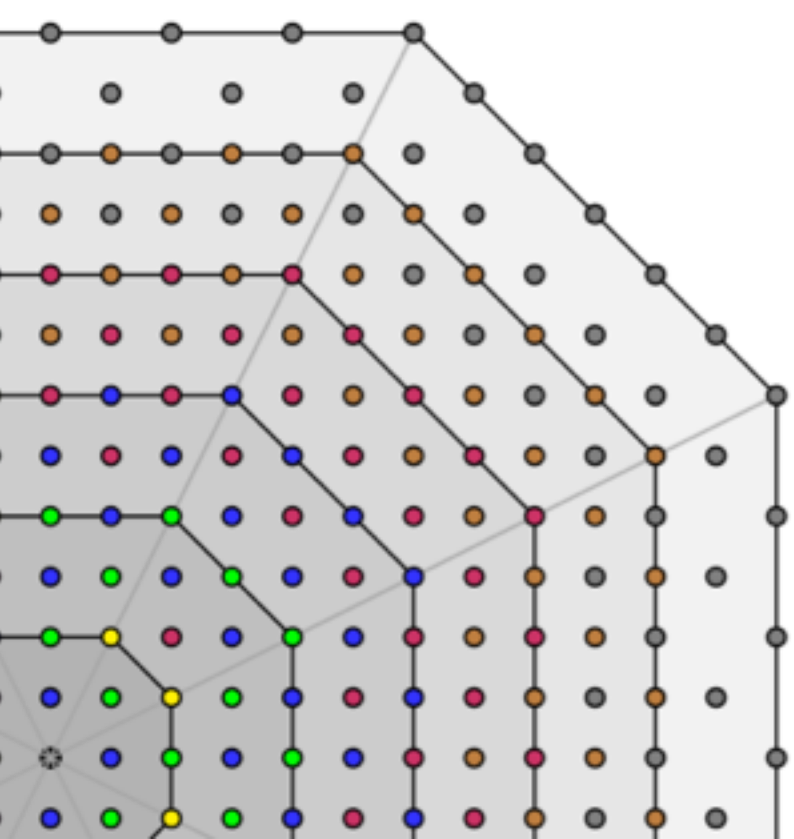

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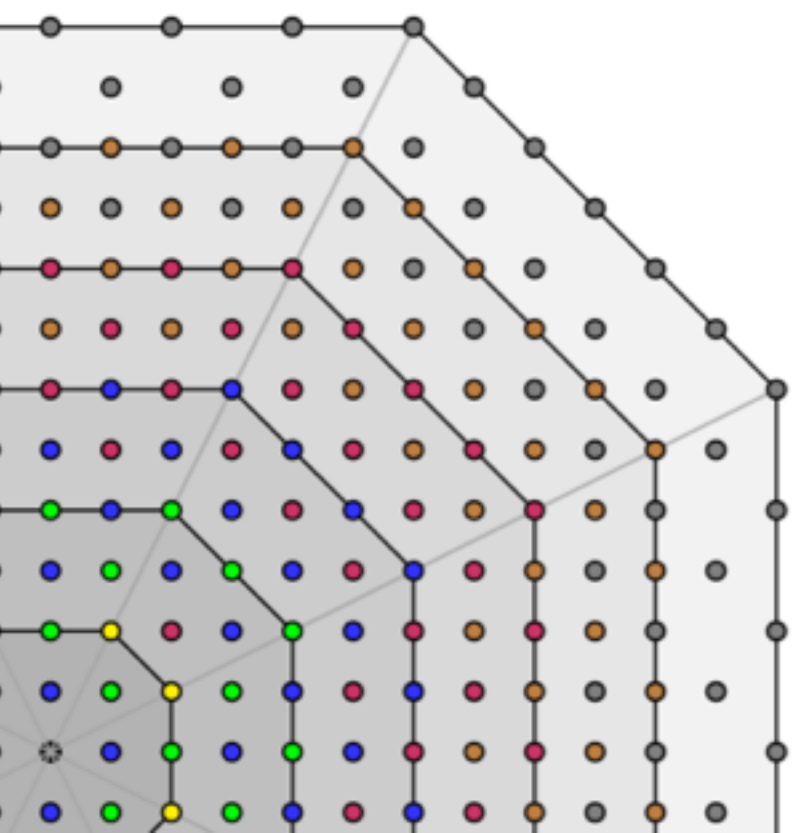

- For that sector, 3 patterns suffice:
  - $a_1^* a_2^*$
  - $a_3\, a_8\, a_1^* a_2^*$
  - $a_3^2\, a_8^2\, a_1^* a_2^*$

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- Here, $G_w(n) = \{(n-1, 2n-3), \ldots, (2n-3, n-1)\}$ for $n \geq 2$.

MAJOR TOOL: COUNTING IN POLYHEDRA

- Let an elementary family $\{E(n)\}$ in $\mathbb{Z}^d$ be defined by finitely many equalities, inequalities, and congruences as below, where the b are affine in n.

$$\begin{cases} \mathsf{a}_i \cdot \mathsf{x} = b_i(n) \ ; \\ \mathsf{a}_j \cdot \mathsf{x} \leq b_j(n) \ ; \\ \mathsf{a}_k \cdot \mathsf{x} \equiv b_k(n) \pmod{c_k} \end{cases}$$

- A **bounded polyhedral family** $\{P(n)\}$ is a finite union of finite intersections of these in which each $P(n)$ is bounded.

$G_W(n)$ from chess example:

$$\begin{cases} x+y=3n+1; \\ x,\ y,\ y\text{-}x/2,\ 2x\text{-}y \ \text{ all } \geq 0 \end{cases}$$

- **Theorem** (Benson): *if f: $\mathbb{Z}^d \to \mathbb{Z}$ is polynomial and $\{P(n)\}$ is a bounded polyhedral family, then*

$$F(x) = \sum_{n=0}^{\infty} \sum_{\mathsf{v} \in P(n)} f(\mathsf{v})\, x^n$$ *is a rational function.*

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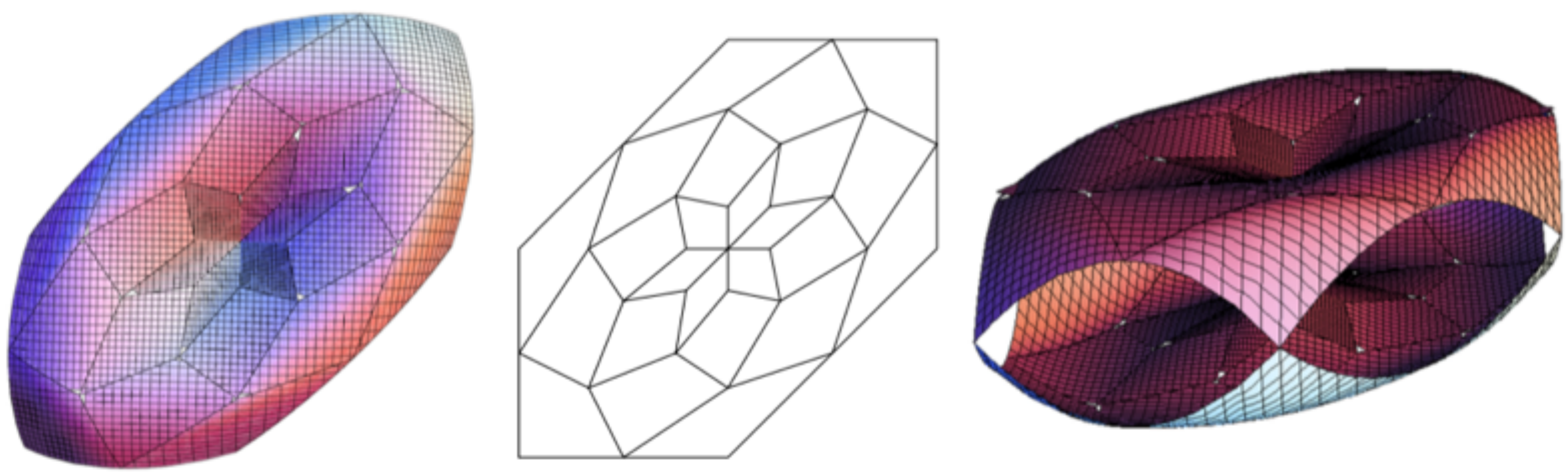

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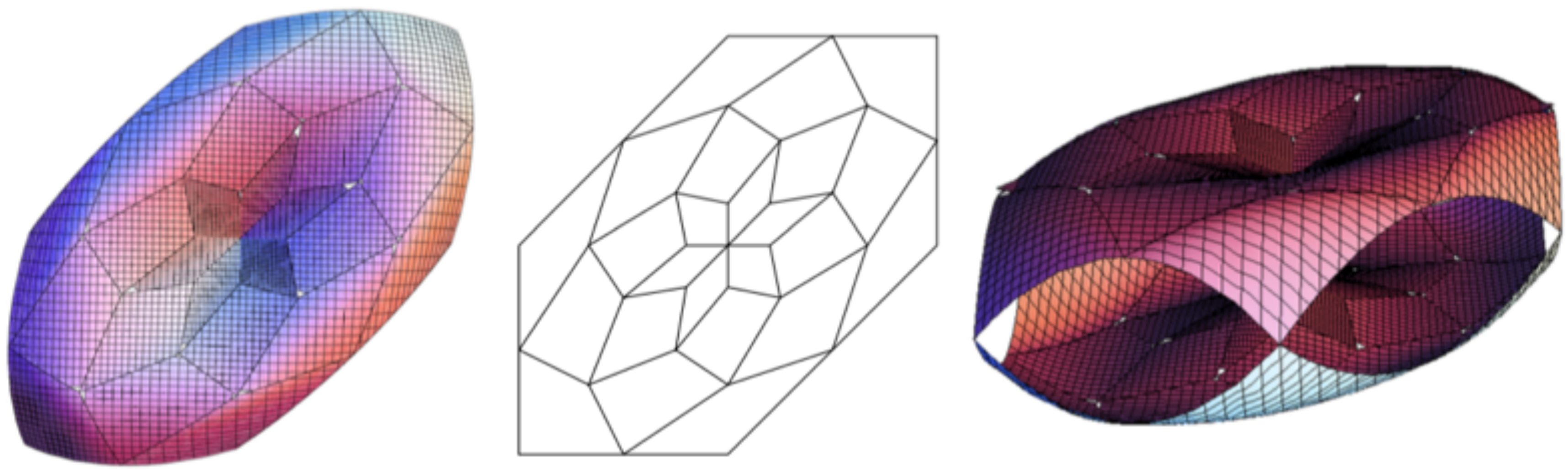

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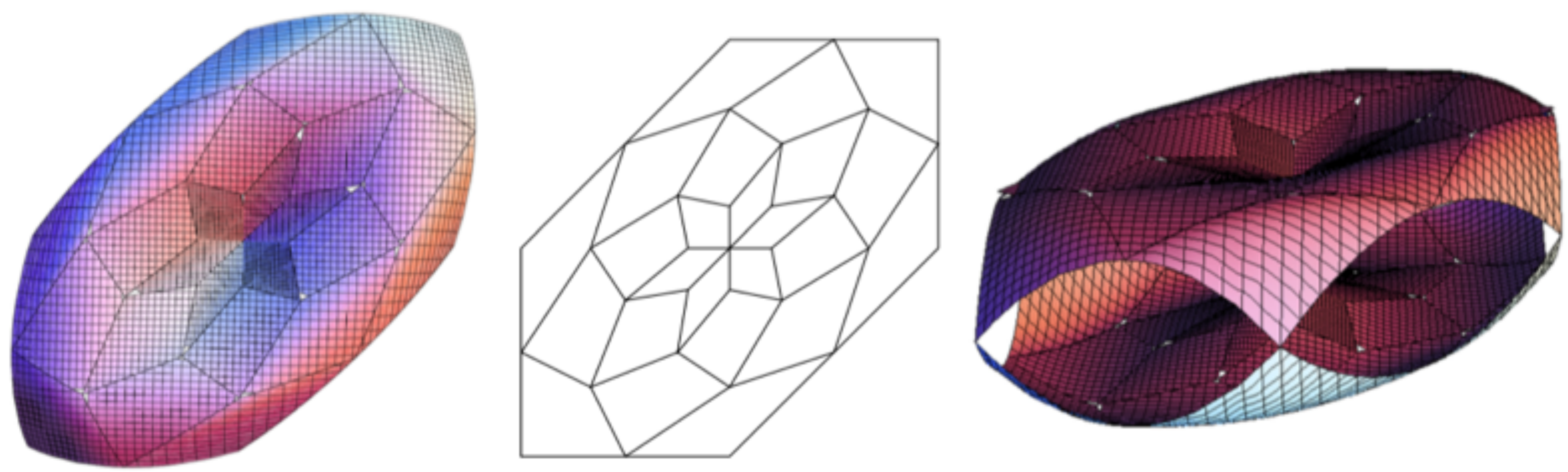

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- **Competition Lemma**: a winning shape or pattern for each (a,b) position is determined by finitely many linear equalities, inequalities, and congruences.

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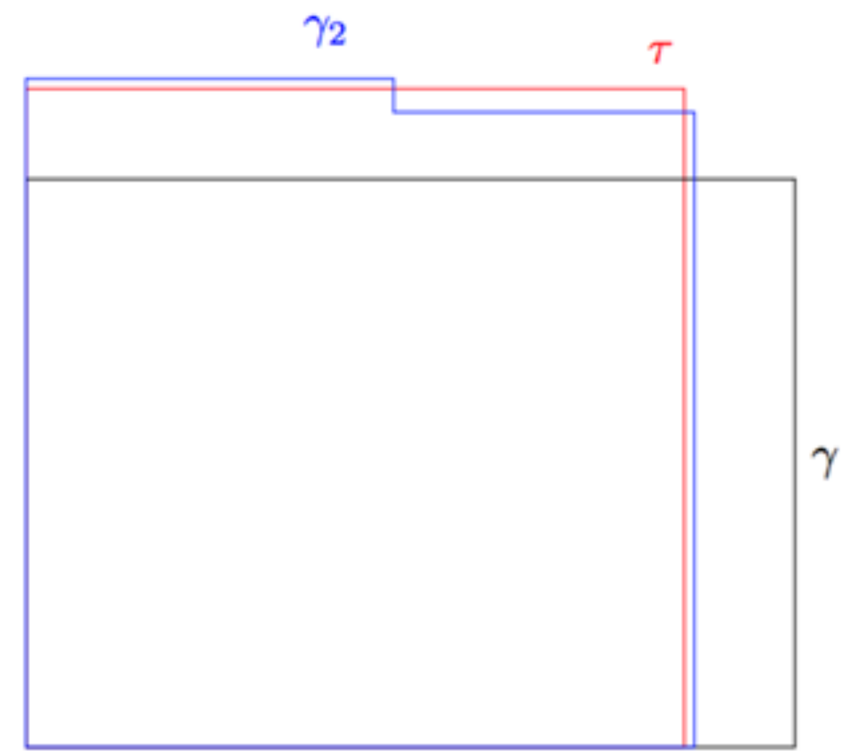

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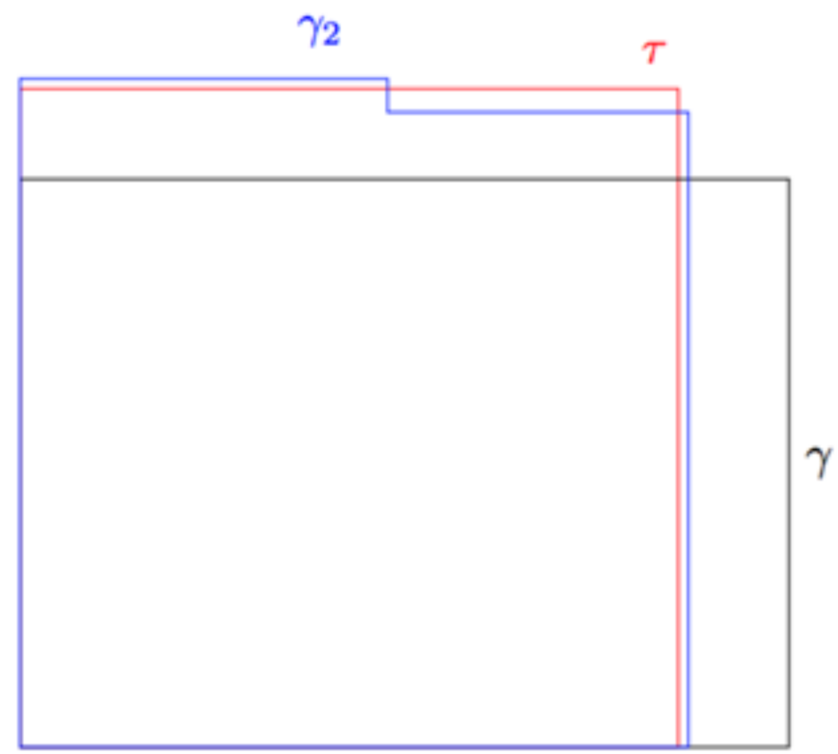

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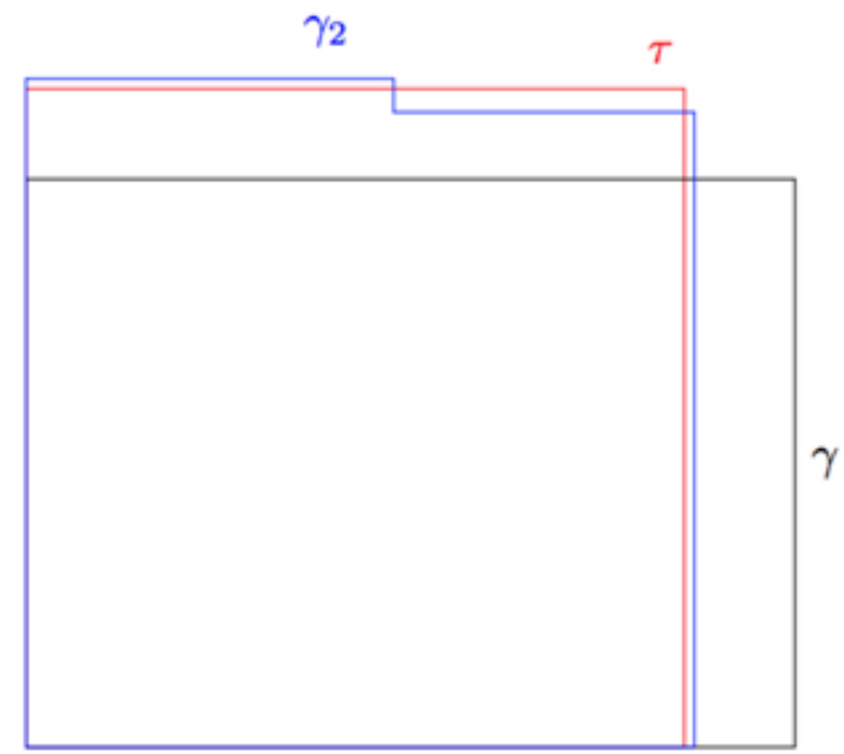

- The length and *(a,b)* position of a shape are affine functions of the inputs, but the height (*c* coordinate) is **quadratic**.
- This is a big problem for writing down which group elements are represented by a shape: the rational competition doesn't allow you to check a quadratic equation. Cute idea to get around this: when two shapes compete, the *difference* in their heights is a linear polynomial, so you ascertain that they reach the same height by checking *linear=0*.

ASSEMBLING THE INGREDIENTS

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GEOME
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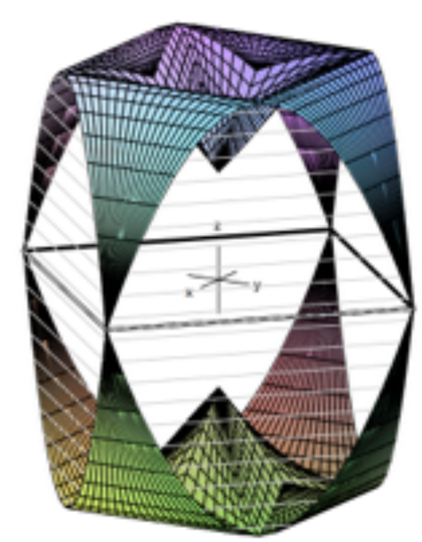

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- We get: **all nilpotent groups have $E<2$**. Proof: CC sphere carries a limit measure that is absolutely continuous with Lebesgue, so there are positive-measure patches with $d(x,y)$ bounded away from $2n$.

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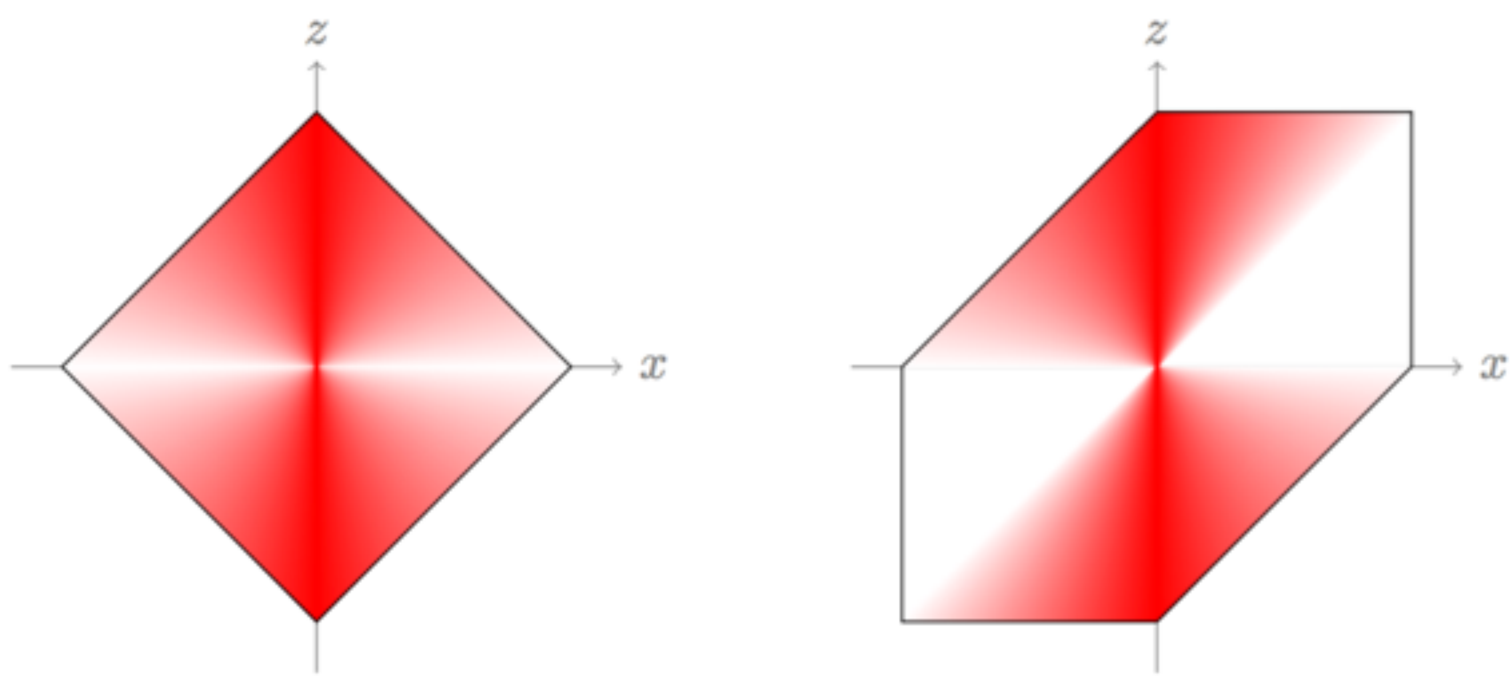

FIGURE 6. The distortion profiles for $K = \{(*, 0, *)\}$ in $H(\mathbb{Z})$ with two different generating sets. The value $d = 1$ is plotted as white, going red as $d \to \infty$.

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- (But 25 pages of combinatorics can be replaced with a quick hit of CC geometry.)

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- Proof: use Mal'cev coordinates to reduce solvability to solving a quadratic equation over a lattice. OTOH, show that systems can encode arbitrary polynomials and quote Hilbert's 10th problem!

MERCI